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Prove that:

$$\sum_{x=2}^{\infty}\frac{(-1)^x\sin^2(x-1)\cos^2(x-1)}{(x-1)^2}=\frac{(\pi-2)^2}{8}$$

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$$\Omega=\sum_{n=2}^{\infty}\frac{(-1)^n\sin^2(n-1)\cos^2(n-1)}{(n-1)^2}$$

$$=-\frac{1}{4}\sum_{n=2}^{\infty}\frac{(-1)^n\sin^2(2n)}{n^2}=\frac{1}{16}\sum_{n=2}^{\infty}\frac{(-1)^n(e^{2in}-e^{-2in})^2}{n^2}=$$

$$=\frac{1}{16}\sum_{n=1}^{\infty}\frac{(-1)^n(e^{4in}-2+e^{-4in})}{n^2}=\frac{1}{16}\sum_{n=2}^{\infty}\frac{(-1)^n(e^{4in}+e^{-4in})}{n^2}-\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=$$

We know $\rightarrow \mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{1}{z}\right)=-\frac{\pi^2}{6}-\frac{\log(-z)}{2}=\frac{1}{16}\left(\mathrm{Li}_2(-e^{4i})+\mathrm{Li}_2\left(-\frac{1}{e^{4i}}\right)\right)+\frac{\pi^2}{96}=$

$$=\frac{1}{16}\left(-\frac{\pi^2}{6}-\frac{\log^2(e^{4i})}{2}\right)+\frac{\pi^2}{96}=-\frac{\pi^2}{96}+\frac{\pi^2}{96}-\frac{\log^2\big(\cos(4)-i\sin(4)\big)}{32}$$

$$=\frac{\log^2\big(\cos(2\pi-4)-i\sin(2\pi-4)\big)}{32}=$$

$$=\frac{\log^2(e^{2\pi-4i})}{32}=\frac{(2\pi-4i)^2}{32}=\frac{(\pi-2)^2}{8}$$