ROMANIAN MATHEMATICAL MAGAZINE

If $f(x)=\sum_{k=0}^{x}\frac{k}{2k+e^2}$ Prove that

$$\int_0^1 f(1-x)dx=\frac{1}{4}-\frac{e^2}{4}\left(\log\left(\frac{e^2}{2}+1\right)-\psi^{(0)}\left(\frac{e^2}{2}+1\right)\right)$$

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$$f(x)=\sum_{k=0}^{x}\frac{k}{2k+e^2}=\frac{1}{2}\sum_{k=0}^{x}\frac{k+\frac{e^2}{2}-\frac{e^2}{2}}{k+\frac{e^2}{2}}=\frac{1}{2}(\sum_{k=0}^{x}(1-\frac{-\frac{e^2}{2}}{k+\frac{e^2}{2}}))=\frac{1}{2}(x+1-\frac{e^2}{2}\sum_{k=0}^{x}\frac{1}{k+\frac{e^2}{2}})=$$

$$=\frac{1}{2}(x+1-\frac{e^2}{2}\left(\psi\left(x+\frac{e^2}{2}+1\right)-\psi\left(\frac{e^2}{2}\right)\right)$$

$$=\frac{1}{2}(x+1-\frac{e^2}{2}\left(\psi\left(x+\frac{e^2}{2}+1\right)-\psi\left(\frac{e^2}{2}+1\right)++\frac{e^2}{2}\right)$$

$$=\frac{1}{2}(x-\frac{e^2}{2}\psi\left(x+\frac{e^2}{2}+1\right)+\psi\left(\frac{e^2}{2}+1\right)+\frac{e^2}{2}$$

$$\Omega=\int_0^1 f(1-x)dx=|1-x=x|=\int_0^1 f(x)dx=\frac{1}{2}\int_0^1 xdx-\frac{e^2}{4}\int_0^1\psi\left(x+\frac{e^2}{2}+1\right)dx+$$

$$+\frac{e^2}{2}\psi\left(\frac{e^2}{2}+1\right)\int_0^1 dx=\frac{1}{4}-\frac{e^2}{4}\left[\log\Gamma(x+\frac{e^2}{2}+1)\right]_0^1+\frac{e^2}{2}\psi\left(\frac{e^2}{2}+1\right)$$

$$=\frac{1}{4}-\log\left(\frac{\Gamma\left(2+\frac{e^2}{2}\right)}{\Gamma\left(1+\frac{e^2}{2}\right)}\right)+\frac{e^2\psi\left(\frac{e^2}{2}+1\right)}{4}$$

Notes: $\psi(x+n+1)-\psi(x)=\sum_{k=0}^{x}\frac{k}{k+x}$; $\psi(x+1)-\psi(x)=\frac{1}{x}$

answer: $\int_0^1 f(1-x)dx=\frac{1}{4}-\frac{e^2}{4}\left(\log\left(\frac{e^2}{2}+1\right)-\frac{e^2}{4}\psi\left(\frac{e^2}{2}+1\right)\right)$