ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$X=\int_0^1 \frac{ln^2(x+1)}{(x+1)(x+3)}dx \quad Y=\int_0^1 \frac{xln^2(x+1)}{(x+1)(x+3)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$X+Y=\int_0^1 \frac{ln^2(x+1)}{x+3}dx \overset{x=2t-1}{\cong} \int_{\frac{1}{2}}^1 \frac{ln^2(2t)}{t+1}dx=$$

$$=\left([\ln(1+t)\,ln^2(2t)+2\ln(2t)\,Li_2(-t)-2Li_3(-t)]\right)_{\frac{1}{2}}^{1}$$

$$X+Y=ln^3(2)+2ln(2)Li_2(-1)-2Li_3(-1)+2Li_3(-\tfrac{1}{2})=ln^3(2)-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)+2Li_3\left(-\frac{1}{2}\right)$$

$$X-Y=\int_0^1 \frac{(1-x)ln^2(x+1)}{(x+1)(x+3)}dx=\int_0^1 \frac{ln^2(x+1)}{x+1}dx-2\int_0^1 \frac{ln^2(x+1)}{x+3}dx$$

$$X-Y=\frac{1}{3}ln^3(2)-2ln^3(2)+\frac{\pi^2}{3}\ln(2)-3\zeta(3)-4Li_3(-\tfrac{1}{2})$$

$$X-Y=-\frac{5}{3}ln^3(2)+\frac{\pi^2}{3}\ln(2)-3\zeta(3)-4Li_3(-\tfrac{1}{2})$$

$$X+Y=ln^3(2)-\frac{\pi^2}{6}\ln(2)+\frac{3}{2}\zeta(3)+2Li_3(-\tfrac{1}{2})$$

$$X=-\frac{1}{3}ln^3(2)+\frac{\pi^2}{12}\ln(2)-\frac{3}{4}\zeta(3)-Li_3(-\tfrac{1}{2})$$

$$Y=\frac{4}{3}ln^3(2)-\frac{\pi^2}{4}\ln(2)+\frac{9}{4}\zeta(3)+3Li_3(-\tfrac{1}{2})$$

We have also : $Li_3\left(-\frac{1}{2}\right)=\frac{1}{4}Li_3(-\frac{1}{4})-\frac{7}{8}\zeta(3)-\frac{1}{6}ln^3(2)+\frac{\pi^2}{12}\ln(2)$

$$\int_0^1 \frac{ln^2(x+1)}{(x+1)(x+3)}dx=\frac{1}{24}\left(-Li_3(\tfrac{1}{4})+3\zeta(3)-4ln^3(2)\right)$$

$$\int_0^1 \frac{xln^2(x+1)}{(x+1)(x+3)}dx=3Li_3\left(-\frac{1}{2}\right)+\frac{9}{4}\zeta(3)+\frac{4}{3}ln^3(2)-\frac{\pi^2}{4}\ln(2)$$