ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^1 \frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1 \frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}dx=\frac{1}{2}\int_0^1 \frac{Li_2(x)\ln(1+x)}{x(1+x)}dx=\frac{1}{2}\underbrace{\int_0^1 \frac{Li_2(x)\ln(1+x)}{x}dx}_{A}-\frac{1}{2}\underbrace{\int_0^1 \frac{Li_2(x)\ln(1+x)}{1+x}}_{B}$$

$$A=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}Li_2(x)dx=\int_0^1 x^{n-1}Li_2(x)=\left(\left[\frac{1}{n}x^nLi_2(x)\right]\right)_0^1+\frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\qquad A=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}-\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{5}{4}\zeta(4)+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}=2Li_4\left(\frac{1}{2}\right)-\frac{11}{4}\zeta(4)+\frac{7}{4}\zeta(3)\ln(2)-\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{12}ln^4(2)$$

$$A=2Li_4\left(\frac{1}{2}\right)-\frac{3}{2}\zeta(4)+\frac{7}{4}\zeta(3)\ln(2)-\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{12}ln^4(2)\,,\; B=\left[\frac{1}{2}Li_2(x)ln^2(1+x)\right]_0^1+\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(1+x)}{x}dx=\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(1+x)}{x}dx\quad ab^2=\frac{1}{6}(a+b)^3+\frac{1}{6}(a-b)^3-\frac{a^2}{3}$$

$$\text{Ln}(1-x)ln^2(1+x)=\frac{1}{6}ln^3(1-x^2)+\frac{1}{6}ln^3\left(\frac{1-x}{1+x}\right)-\frac{1}{3}ln^3(1-x)$$

$$B=\frac{1}{2}\zeta(2)ln^2(2)+\frac{1}{12}\underbrace{\int_0^1\frac{ln^3(1-x^2)}{x}dx}_{1-x^2\to x}+\frac{1}{12}\underbrace{\int_0^1\frac{\left(\frac{1-x}{1+x}\right)}{x}dx}_{\frac{1-x}{1+x}\to x}-\frac{1}{6}\underbrace{\int_0^1\frac{ln^3(1-x)}{x}dx}_{1-x\to x}=\frac{1}{2}\zeta(2)ln^2(2)+$$

$$\frac{1}{6}\sum_{n=1}^{\infty}\int_0^1 x^{2n-2}ln^3(x)dx-\frac{1}{8}\sum_{n=1}^{\infty}\int_0^1 x^{n-1}ln^3(x)dx=\frac{1}{2}\zeta(2)ln^2(2)-\frac{3}{16}\zeta(4)$$

$$\Omega=Li_4\left(\frac{1}{2}\right)+\frac{7}{8}\zeta(3)\ln(2)-\frac{1}{2}\zeta(2)ln^2(2)-\frac{21}{32}\zeta(4)+\frac{1}{24}ln^4(2)$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1 x^{n-1}Li_2(x)dx=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^nH_n\left[\frac{H_n}{n^2}-\frac{\zeta(2)}{n}\right]=\frac{1}{2}(\Omega_1-\Omega_2)$$

$$\Omega_1=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^2}{n^2}=-\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{n-1}+\frac{1}{n}\right)}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx=$$

$$-\int_0^1\frac{\ln(1-x)}{x}\left(\sum_{n=1}^{\infty}\frac{(-1)^nH_{n-1}x^n}{n}+\sum_{n=1}^{\infty}\frac{(-1)^nx^n}{n^2}\right)dx=-\frac{1}{2}\int_0^1\frac{\ln(1-x)}{x}ln^2(1+x^2)dx+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}=J-K$$

$$J=\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^{n-1}ln^2(x)dx=-$$

$$\frac{1}{2}\int_0^1\frac{\ln(1+x)ln^2(x)}{x(1+x)}dx=\frac{1}{2}\Bigg(\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-x)ln^2(x)}{x}dx+\int_0^{\frac{1}{2}}\frac{ln^3(1-x)}{x}dx}_{J_1}-\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(x)ln^2(1-x)}{x}dx}_{J_2}=0.5J_1-J_2$$

$$J_1=\int_0^{\frac{1}{2}}\frac{ln^2(x)\ln(1-x)}{x}dx+\int_0^{\frac{1}{2}}\frac{ln^3(1-x)}{x}dx=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{1}{2}}x^{n-1}ln^2(x)dx+\int_{\frac{1}{2}}^1\frac{ln^3(x)}{1-x}dx$$

$$=\frac{1}{3}ln^4(2)+\frac{1}{3}\int_0^{\frac{1}{2}}\frac{ln^3(x)}{1-x}dx+\int_0^1\frac{ln^3(x)}{1-x}dx+\int_{\frac{1}{2}}^0\frac{ln^3(x)}{1-x}dx=4Li_4\left(\frac{1}{2}\right)-6\zeta(4)+\frac{7}{2}\zeta(3)\ln(2)-\zeta(2)ln^2(2)+\frac{2}{3}ln^4(2)$$

$$J_2=\int_0^{\frac{1}{2}}\frac{\ln(x)ln^2(1-x)}{x}dx=\frac{1}{4}ln^4(2)+\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(x)}{1-x}dx$$

$$=\frac{1}{4}ln^4(2)-\frac{1}{2}\sum_{n=1}^{\infty}H_n\int_0^1 x^n ln^2(x)dx=\frac{1}{4}ln^4(2)+2\zeta(4)-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}$$

$$=\frac{1}{4}ln^4(2)+2\zeta(4)-\frac{9}{4}\zeta(4)=\frac{1}{4}ln^4(2)-\frac{\pi^4}{360}$$

$$J=\frac{1}{2}J_1-J_2=2Li_4\left(\frac{1}{2}\right)+\frac{7}{4}\ln(2)\zeta(3)+\frac{ln^4(2)}{12}-\frac{\pi^2}{12}ln^2(2)-\frac{11\pi^4}{360}$$

$$K=-\frac{1}{2}\underbrace{\int_0^1\frac{\ln(1-x)ln^2(1+x)}{x}dx}_{ab^2=>\frac{1}{6}(a+b)^3+\frac{1}{6}(a-b)^3-\frac{a^3}{3}}=\frac{1}{8}\int_0^1\frac{ln^3(x)}{1-x}dx-\frac{1}{12}\underbrace{\int_0^1\left(ln^3\left(\frac{1-x}{1+x}\right)\right)/xdx}_{\frac{1-x}{1+x}=>x}=$$

$$\frac{1}{8}\sum_{n=1}^{\infty}\int_0^1 x^n ln^3(x)dx-\frac{1}{6}\int_0^1\frac{ln^3(x)}{1-x^2}dx=-\frac{\pi^4}{480}$$

$$\Omega_1=J-K=2Li_4\left(\frac{1}{2}\right)+\frac{7}{4}\zeta(3)\ln(2)+\frac{1}{12}ln^4(2)-\frac{\pi^2}{12}ln^2(2)-\frac{41\pi^4}{1440}$$

$$\Omega_2=\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n}=\zeta(2)\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^{n-1}dx=$$

$$-\zeta(2)\underbrace{\int_0^1\frac{\ln(1+x)}{x(1+x)}dx}_{\left\{\frac{1}{x(1+x)}=\frac{1}{x}-\frac{1}{1+x}\right\}}=\frac{1}{2}\zeta(2)ln^2(2)-\frac{\zeta^2(2)}{2}=\frac{\pi^2}{12}ln^2(2)-\frac{\pi^4}{72}$$

Answer: $\Omega=\int_0^1\frac{Li_2(x^2)\ln(1+x^2)}{x(1+x^2)}dx=\frac{1}{2}(\Omega_1-\Omega_2)=Li_4\left(\frac{1}{2}\right)+\frac{7}{8}\zeta(3)\ln(2)+\frac{1}{24}ln^4(2)-\frac{\pi^2}{12}ln^2(2)-\frac{7\pi^4}{960}$

Note : ζ(3) --- Apery's constant