ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^1\frac{\sqrt{x}\ln(\ln(x))}{1+x}dx$$

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$$\Omega=\int_0^1\frac{\sqrt{x}\ln(\ln(x))}{1+x}dx\overset{x\to x^2}{\cong}2\int_0^1\frac{x^2\ln(2\ln(x))}{1+x^2}dx=2\ln(-2)\int_0^1\frac{x^2}{1+x^2}dx+2\int_0^1\frac{x^2\ln(-\ln(x))}{1+x^2}dx$$

$$\int_0^1\frac{x^2}{1+x^2}dx=\int_0^1\left(1-\frac{1}{1+x^2}\right)dx=1-\frac{\pi}{4}\ ,\ \ln(-2)=2\ln(i)+\ln(2)=i\pi+\ln(2)$$

$$\Omega=\ln(4)-\frac{\pi}{2}\ln(2)-\frac{1}{2}i\pi(\pi-4)+2\int_0^1\ln(-\ln(x))\,dx-2\int_0^1\frac{\ln(-\ln(X))}{1+x^2}dx$$

$$\int_0^1\ln(-\ln(x))\,dx\overset{x\to e^{-t}}{\cong}\int_0^\infty\ln(t)\,e^{-t}dt=-\gamma$$

Malmsten's integral: $M(\varphi)=\int_0^1\frac{\ln(-\ln(x))}{1+2x\cos(\varphi)}dx=\frac{\pi}{2\sin(\varphi)}ln\left\{\frac{(2\pi)^{\frac{\varphi}{\pi}}\Gamma\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\varphi}{2\pi}\right)}\right\}\quad -\pi<\varphi<\pi$

$$M\left(\frac{\pi}{2}\right)=\int_0^1\frac{\ln(-\ln(X))}{1+x^2}dx=\frac{\pi}{2}ln\left\{\frac{\sqrt{2\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right\}=\frac{\pi}{4}\ln(2)+\frac{\pi}{4}\ln(\pi)-\pi ln\Gamma\left(\frac{1}{4}\right)+\frac{\pi}{2}ln\underbrace{\left\{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\right\}}_{\pi\sqrt{2}}$$

$$\int_0^1\frac{\ln(-\ln(x))}{1+x^2}dx=\frac{\pi}{2}\ln(2)+\frac{3\pi}{4}\ln(\pi)-\pi ln\Gamma\left(\frac{1}{4}\right)$$

$$\Omega=\ln(4)-\frac{\pi}{2}\ln(2)-\frac{1}{2}i\pi(\pi-4)-2\gamma-\pi\ln(2)-\frac{3\pi}{2}\ln(\pi)+2\pi\ln\left(\Gamma\left(\frac{1}{4}\right)\right)$$

$$\int_0^1\frac{\sqrt{x}\ln(\ln(x))}{1+x}dx=\ln(4)+2\pi ln\Gamma\left(\frac{1}{4}\right)-\frac{3\pi}{2}\ln(2\pi)-\frac{1}{2}i\pi(\pi-4)-2\gamma$$