ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^\infty \frac{x\log^2(1+x)}{(1+x)(2+x)^3}dx=\frac{1}{6}\left(2\pi^2-9\zeta(3)-6\log(4)\right)$$

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Let x+1=y

$$I=\int_1^\infty \frac{(y-1)\log^2(y)}{y(1+y)^3}dy\Bigg]_{\frac{1}{y}=z}=\int_0^1 \frac{z(1-z)\log^2(z)}{(1+z)^3}dz$$

$$=\int_0^1 \frac{3(1+z)-(1+z)^2-2}{(1+z)^3}\log^2(z)\,dz$$

$$=3\int_0^1 \frac{1}{(1+z)^2}\log^2(z)\,dz-\int_0^1 \frac{1}{1+z}\log^2(z)\,dz-2\int_0^1 \frac{1}{(1+z)^3}\log^2(z)\,dz$$

$$=3\int_0^1 \left\{-\sum_{k=0}^\infty(-1)^k k z^{k-1}\right\}\log^2(z)\,dz-\int_0^1 \left\{\sum_{k=0}^\infty(-1)^k z^k\right\}\log^2(z)\,dz$$

$$-2\int_0^1 \left\{\frac{1}{2}\sum_{k=0}^\infty(-1)^k k(k-1)z^{k-2}\right\}\log^2(z)\,dz$$

$$=-3\int_0^1 \left\{\sum_{k=1}^\infty(-1)^k k z^{k-1}\right\}\log^2(z)\,dz-\int_0^1 \left\{\sum_{k=0}^\infty(-1)^k z^k\right\}\log^2(z)\,dz$$

$$-\int_0^1 \left\{\sum_{k=2}^\infty(-1)^k k(k-1)z^{k-2}\right\}\log^2(z)\,dz=$$

$$= -3\sum_{k=1}^{\infty}(-1)^k k\int_0^1 z^{k-1}\log^2(z)\,dz - \sum_{k=0}^{\infty}(-1)^k\int_0^1 z^k\log^2(z)\,dz$$

$$-\sum_{k=2}^{\infty}(-1)^k k(k-1)\int_0^1 z^{k-2}\log^2(z)\,dz$$

$$= -6\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3} - 2\sum_{k=2}^{\infty}\frac{(-1)^k k}{(k-1)^2}$$

$$= -6\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3} - 2\sum_{k=2}^{\infty}\frac{(-1)^k}{k-1} - 2\sum_{k=2}^{\infty}\frac{(-1)^k}{(k-1)^2}$$

$$= 6\eta(2) - 2\eta(3) - 2\log(2) - 2\eta(2) = 4\eta(2) - 2\eta(3) - 2\log(2)$$

$$= 2\zeta(2) - \frac{3}{2}\zeta(3) - 2\log(2) = \frac{1}{6}\left(2\pi^2 - 9\zeta(3) - 6\log(4)\right)$$