ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\Omega=\int_0^1 \frac{\log^2(x)}{(1+x+x^2)(1+x+x^2+x^3)}dx=\frac{21}{32}\zeta(3)+\frac{8\pi^3}{81\sqrt{3}}-\frac{\pi^3}{32}$$

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Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1 \frac{\log^2(x)}{(1+x+x^2)(1+x+x^2+x^3)}dx=\int_0^1 \frac{\log^2(x)}{1+x+x^2}dx-\int_0^1 \frac{x\log^2(x)}{1+x+x^2+x^3}dx=$$

$$\underbrace{\int_0^1 \frac{(1-x)\log^2(x)}{1-x^3}dx}_{A}-\underbrace{\int_0^1 \frac{x(1-x)\log^2(x)}{1-x^4}dx}_{B}$$

$$A\overset{x^3=>x}{\cong}\frac{1}{27}\int_0^1 \frac{(x^{\frac{1}{3}-1}-x^{\frac{2}{3}-1})\log^2(x)}{1-x}dx=\frac{1}{27}\left(\psi^{(2)}\left(\frac{2}{3}\right)-\psi^{(2)}\left(\frac{1}{3}\right)\right)$$

$$A=\frac{\pi}{27}\lim_{Z\to\frac{1}{3}}\frac{d^2}{z^2d}\cot(\pi z)=\frac{2\pi^3}{27}\lim_{z\to\frac{1}{3}}\frac{\cot(\pi z)}{\sin^2(\pi z)}=\frac{8\pi^3}{81\sqrt{3}}$$

$$B\overset{x^4\to x}{\cong}\frac{1}{64}\int_0^1 \frac{(x^{\frac{1}{2}-1}-x^{\frac{3}{4}-1})\log^2(x)}{1-x}dx=\frac{1}{64}\left(\psi^{(2)}\left(\frac{3}{4}\right)-\psi^{(2)}\left(\frac{1}{2}\right)\right)$$

$$B=\frac{1}{64}\left(2\pi^3-42\zeta(3)\right)=\frac{\pi^3}{32}-\frac{21}{32}\zeta(3)$$