ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_1^{\infty}\frac{\ln(1+x)\ln(1+x^2)}{(1+x)^2}dx$$

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$$I=\int_1^{\infty}\frac{\ln(1+x)\ln(x^2+1)}{(1+x)^2}dx$$
$$=\frac{1}{2}\int_1^{\infty}\ln(1+x)\ln\left(1+x^2\right)d\left(\frac{x-1}{x+1}\right), t=\frac{x-1}{x+1}=>x=\frac{1+t}{1-t}$$
$$I=\frac{1}{2}\int_0^1(\ln(2)-\ln(1-t))\left(\ln(2)+\ln\left(t^2+1\right)-2\ln(1-t)\right)dt=$$
$$\frac{1}{2}\int_0^1\left(ln^2(2)+\ln(2)\ln\left(t^2+1\right)-3\ln(2)\ln(1-t)-\ln(1-t)\ln\left(1+t^2\right)+2ln^2(1-t)\right)dt$$
$$=$$

By integration by parts, we easily find:

$$\int\ln\left(1+x^2\right)dx=x\left(\ln\left(1+x^2\right)-2\right)+tan^{-1}(x)+C$$

$$\int ln^2(1-x)dx=2x+(x-1)\ln(1-x)\,(\ln(1-\mathrm{x})-2)+\mathrm{C}$$

$$I=\frac{1}{2}ln^2(2)+\frac{1}{2}ln^2(2)-\ln(2)+\frac{1}{4}\pi\ln(2)+\frac{3}{2}\ln(2)-\frac{1}{2}\int_0^1\ln(1-x)\ln\left(1+x^2\right)dx+2=$$

$$=ln^2(2)+\frac{1}{2}\ln(2)+\frac{1}{4}\pi\ln(2)-\frac{1}{2}\int_0^1\ln(1-x)\ln\left(1+x^2\right)dx+2$$

$$J=\int_0^1\ln(1-x)\ln\left(1+x^2\right)dx=-\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\int_0^1x^{2n}\ln(1-x)\,dx=$$

$$=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\frac{H_{2n+1}}{2n+1}=\sum_{n=1}^{\infty}(-1)^nH_{2n+1}\left(\frac{1}{n}-\frac{2}{2n+1}\right)=$$

$$-\sum_{n=1}^{\infty}(-1)^n\left(H_{2n}+\frac{1}{2n+1}\right)\left(\frac{1}{n}-\frac{2}{2n+1}\right)=\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{n}-\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n+1}-\left(2G+\ln(2)+\frac{\pi}{2}-4\right)=$$

$$2\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n}-2\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n+1}-\left(2G+\ln(2)+\frac{\pi}{2}-4\right)=$$

$$2\Re\sum_{n=1}^{\infty} i^n \frac{H_n}{n} - 2\sum_{n=1}^{\infty} (-1)^n \frac{H_{2n}}{2n+1} x^{2n+1}]_{x=1} - \left(2G + \ln(2) + \frac{\pi}{2} - 4\right) =$$

$$2\Re(Li_2(i) + \frac{1}{2} ln^2(1-i)) + tan^{-1}(x)\ln(1+x^2)]_{x=1} - (2G + \ln(2) + \frac{\pi}{2} - 4) =$$

$$2(-\frac{5\pi^2}{96} + \frac{1}{8} ln^2(2)) + \frac{\pi}{4}\ln(2) - 2G - \ln(2) - \frac{\pi}{2} + 4 =$$

$$-\frac{5\pi^2}{48} + \frac{1}{4} ln^2(2) + \frac{\pi}{4}\ln(2) - 2G - \ln(2) - \frac{\pi}{2} + 4$$

$$I = ln^2(2) + \frac{1}{2}\ln(2) + \frac{1}{4}\pi\ln(2) + 2 - \frac{1}{2}(-\frac{5\pi^2}{48} + \frac{1}{4} ln^2(2) + \frac{\pi}{4}\ln(2) - 2G - \ln(2) - \frac{\pi}{2} + 4)$$

$$= \frac{7}{8} ln^2(2) + G + \frac{5\pi^2}{96} + \frac{\pi}{8}(2 + \ln(2)) + \ln(2)$$

Note : G --> Catalan's constant