ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \int_0^1 \frac{\ln(xy)\ln(1+x^2)}{x(1+y^2)^2}\,dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Gbenga Ajeigbe-Nigeria

$$I = \int_0^1 \int_0^1 \frac{\ln(xy)\ln(1+x^2)}{x(1+y^2)^2}\,dxdy$$

$$= \int_0^1 \int_0^1 \frac{\ln(x)\ln(1+x^2)}{x(1+y^2)^2}\,dxdy + \int_0^1 \int_0^1 \frac{\ln(y)\ln(1+x^2)}{x(1+y^2)^2}\,dxdy$$

$$I = \int_0^1 \frac{dy}{(1+y^2)^2} \int_0^1 \frac{\ln(x)\ln(1+x^2)}{x}\,dx + \int_0^1 \frac{\ln(y)}{(1+y^2)^2}\,dy \int_0^1 \frac{\ln(1+x^2)}{x}\,dx$$

$$I = \frac{1}{2}[\frac{y}{y^2+1} + \arctan(x)]_0^1 \left[-\frac{3\zeta(3)}{4}\right] + ([-\frac{G}{2} - \frac{1}{8}][-\frac{1}{2}Li_2(\text{-}1)]$$

$$[\frac{1}{4} + \frac{\pi}{8}]\left[-\frac{3\zeta(3)}{4}\right] + \left[-\frac{G}{2} - \frac{\pi}{8}\right][\frac{\pi^2}{24}]$$

$$I = -\frac{3\zeta(3)}{16} - \frac{3\zeta(3)}{32}\pi - \pi^2\frac{G}{48} - \frac{\pi^3}{192}$$

$$I = \frac{1}{384}[-8\pi^2 G - 9\pi\zeta(3) - 18\zeta(3) - 2\pi^3]$$

Solution 2 by Kartick Chandra Betal-India

$$\int_0^1 \int_0^1 \frac{\ln(xy)\ln(1+x^2)}{x(1+y^2)^2}\,dxdy$$

$$= \int_0^1 \frac{\ln(x)\ln(1+x^2)}{x}\,dx \int_0^1 \frac{dy}{(1+y^2)^2}$$

$$+ \int_0^1 \frac{\ln(1+x^2)}{x}\,dx \int_0^1 \frac{\ln(y)}{(1+y^2)^2}\,dy = AB + CD = \left(-\frac{3}{16}\zeta(3)\right)\left(\frac{\pi+2}{8}\right) + \left\{-\left(\frac{G}{2} + \frac{\pi}{8}\right)\right\} =$$

$$\frac{1}{384}\{-9\pi\zeta(3) - 18\zeta(3) - 8G\pi^2 - 2\pi^3\}$$

$$A = \int_0^1 \frac{\ln(x)\ln(1+x^2)}{x}\,dx$$

$$= \frac{1}{4}\int_0^1 \frac{\ln(x)\ln(!+x)}{x}\,dx = \frac{1}{4}[-Li_2(-x)\ln(x)]_0^1 + \frac{1}{4}[Li_3(-x)]_0^1$$

$$= -\frac{3}{16}\zeta(3)$$

ROMANIAN MATHEMATICAL MAGAZINE

$$B=\int_0^1\frac{dy}{(1+y^2)^2}=\int_0^{\frac{\pi}{4}}cos^2(y)dy=\frac{1}{2}[y+\frac{\sin(2y)}{2}]_0^{\frac{\pi}{4}}=(\frac{\pi+2}{8})$$

$$C=\int_0^1\frac{\ln(1+x^2)}{x}dx=\frac{1}{2}\int_0^1\frac{\ln(1+x)}{x}dx=\frac{\pi^2}{24}$$

$$D=\int_0^1\frac{\ln(y)}{(1+y^2)^2}dy=\frac{1}{2}[\left(\arctan(y)+\frac{y}{1+y^2}\right)\ln(y)]_0^1-\frac{1}{2}\int_0^1\frac{\left(\arctan(y)+\frac{y}{1+y^2}\right)}{y}dy$$

$$=-\frac{1}{2}\int_0^1\frac{\arctan(y)}{y}dy-\frac{1}{2}[arctan(y)]_0^1$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln(\tan(y))\,dy-\frac{\pi}{8}=-(\frac{G}{2}+\frac{\pi}{8})$$

Note : $\begin{cases} G \to is\ Catalan's\ constant \\ \zeta(3) \to is\ Apery's\ constant \end{cases}$