ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^1\int_0^1 \ln\left(\sqrt{\frac{x}{x^2+1}+\frac{y}{y^2+1}}\right)dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\frac{1}{2}\int_0^1\int_0^1 \ln\left(\frac{x}{1+x^2}+\frac{y}{y^2+1}\right)dxdx=\frac{1}{2}\int_0^1\int_0^1\left(\frac{x(1+y^2)+y(1+x^2)}{(1+x^2)(1+y^2)}\right)dxdy=\frac{1}{2}(\Omega_1-\Omega_2)$$

$$\Omega_1=\int_0^1\int_0^1 \ln\left(x(1+y^2)+y(1+x^2)\right)dxdy=\int_0^1\int_0^1 \ln(x+xy^2+y+x^2y)\,dxdy=$$

$$\int_0^1\int_0^1 \ln((xy+1)(x+y))\,dxdy=\int_0^1\int_0^1 \ln(1+xy)\,dxdy+\int_0^1\int_0^1 \ln(x+y)\,dxdy=$$

$$\underbrace{-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1\int_0^1 x^ny^n\,dxdy}+\underbrace{\int_0^1\int_0^1 \ln(x+y)\,dxdy}=I_1+I_2$$

$$I_1=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)^2}=-\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2}\right)=-2+2\ln(2)+\frac{\pi^2}{12}$$

$$I_2=\int_0^1\int_0^1 \ln(x+y)\,dxdy=\int_0^1\left((x+y)\ln(x+y)-x\right)\Big|_0^1 dy=$$

$$\int_0^1\left((1+y)\ln(1+y)-y\ln(y)-1\right)dy=\frac{1}{2}\left[(y+1)^2\ln(1+y)-3y-y^2\ln(y)\right]\Big|_0^1=$$

ln(4)-1.5 $\Omega_1=I_1+I_2=\ln(16)+\frac{\pi^2}{12}-3.5$

$$\Omega_2=\int_0^1\int_0^1 \ln(1+x^2)\,dxdy+\int_0^1\int_0^1 \ln(1+y^2)\,dxdy=2\int_0^1\int_0^1 \ln(1+x^2)\,dxdy=$$

$$-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1\int_0^1 x^{2n}dxdy=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n(2n+1)}=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}=$$

$=2\ln(2)+\pi-4$

$$\Omega=\frac{1}{2}(\Omega_1-\Omega_2)=\frac{1}{4}+\frac{\pi^2}{24}+\ln(2)-\frac{\pi}{2}$$