ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx$$

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Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{2n}\ln(1+x^2)\,dx \overset{IBP}{\cong}$$

$$\sum_{n=0}^{\infty}(-1)^n\left[\frac{x^{2n+1}}{2n+1}\ln(1+x^2)\right]_0^1 - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 \frac{x^{2n+1}}{x^2+1}dx =$$

$$\ln(2)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - 2\int_0^1 \frac{xtan^{-1}}{(x^2+1)}dx = \ln(2)\,tan^{-1}(1) - 2\int_0^1 \frac{xtan^{-1}(x)}{(1+x^2)}dx =$$

$$\frac{\pi}{4}\ln(2) - 2\int_0^1 \frac{xtan^{-1}(x)}{1+x^2}dx \overset{\arctan(x)=>t}{\cong} \frac{\pi}{4}\ln(2) - 2\int_0^{\frac{\pi}{4}} ttan(t)dt =$$

$$\frac{\pi}{4}\ln(2) + 2[tlog(\cos(t))]_0^{\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}} \log(\cos(t))\,dt = 2\int_0^{\frac{\pi}{4}} \log(\cos(t))\,dt =$$

$$2\ln(2)\int_0^{\frac{\pi}{4}} dt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nt)\,dt = \frac{\pi}{2}\ln(2) + \sum_{n=1}^{\infty}\frac{(-1)^n\sin(\frac{\pi n}{2})}{n^2} = \frac{\pi}{2}\ln(2) - G$$

$$\Omega=\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx = \frac{\pi}{2}\ln(2) - G$$

Solution 2 by Bui Hong Suc-Vietnam

We have :

$$\begin{Bmatrix} \int_0^{\frac{\pi}{4}}\ln(\tan(x))\,dx = \int_0^{\frac{\pi}{4}}\ln(\sin(x))\,dx - \int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx = -G \\ \int_0^{\frac{\pi}{2}}\ln(\sin(x))\,dx = \int_0^{\frac{\pi}{4}}\ln(\sin(x))\,dx + \int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx = -\frac{\pi}{2}\ln(2) \end{Bmatrix}$$

$$\Omega=\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx \underbrace{=}_{x=\tan(t)} \int_0^{\frac{\pi}{4}} \frac{\ln(\frac{1}{cos^2(t)})}{\frac{1}{cos^2(t)}}\frac{dt}{cos^2(t)} = -2\int_0^{\frac{\pi}{4}}\ln(cos(t))=$$

$$-2(-\frac{\pi}{4}\ln(2) + \frac{G}{2}) = \frac{\pi}{2}\ln(2) - G$$