ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^1\int_0^1\frac{\ln(1-x^2y^2)+xyln(\frac{1-xy}{1+xy})}{1-x^2y^2}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Bui Hong Suc-Vietnam

We have: $\zeta(s)\Gamma(s)=\int_0^\infty\frac{x^{z-1}}{e^x-1}dx\ \ Li_3(1)=\zeta(3)$

$$Li_3\left(\frac{1}{2}\right)=\frac{ln^3(2)}{6}-\frac{\pi^2\ln(2)}{12}+\frac{7\zeta(3)}{8},Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{ln^2(2)}{2}$$

$$\Omega=\frac{\ln(1-x^2y^2)+xyln(\frac{1-xy}{1+xy})}{1-x^2y^2}dxdy=$$

$$\int_0^1\int_0^1\frac{\ln(1-xy)+\ln(1+xy)+xy(\ln(1-xy)-\ln(1+xy))}{(1-xy)(1+xy)}dxdy=$$

$$\int_0^1\int_0^1\frac{\ln(1-xy)}{1-xy}dxdy+\int_0^1\int_0^1\frac{\ln(1+xy)}{1+xy}dxdy=-\int_0^1\frac{dy}{y}\int_0^1\ln(1-xy)\,d(\ln(1-xy))+$$

$$\int_0^1\frac{dy}{y}\int_0^1\ln(1+xy)\,d(\ln(1+xy))=\frac{1}{2}(-\int_0^1\frac{dy}{y}ln^2(1-xy)\Big|_0^1+$$

$$\int_0^1\frac{dy}{y}ln^2(1+xy)\Big|_0^1)=\frac{1}{2}(-\underbrace{\int_0^1\frac{ln^2(1-y)dy}{y}}_{I}+\underbrace{\int_0^1\frac{ln^2(1+y)}{y}dy}_{J}$$

$$I=\int_0^1\frac{ln^2(1-y)dy}{y}\overset{\ln(1-y)=-x}{\triangleq}\int_0^\infty\frac{x^2e^{-x}}{1-e^{-x}}=\int_0^\infty\frac{x^2dx}{e^x-1}=\zeta(3)\Gamma(3)=2\zeta(3)$$

$$J=\int_0^1\frac{ln^2(1+y)}{y}dy\overset{X=\frac{1}{y+1}}{\triangleq}\int_{\frac{1}{2}}^1\frac{ln^2(x)}{\frac{1-x}{x}x^2}dx=\int_{\frac{1}{2}}^1\frac{ln^2(x)}{x(1-x)}dx=\int_{\frac{1}{2}}^1\frac{ln^2(x)}{1-x}dx+\int_{\frac{1}{2}}^1\frac{ln^2(x)}{x}dx=$$

$$(-ln^2(x)\ln(1-x)-2\ln(x)Li_2(x)+2Li_3(x))\Big|_{\frac{1}{2}}^1+\frac{ln^3(x)}{3}\Big|_{\frac{1}{2}}^1=$$

$$2Li_3(1)-2Li_3\left(\frac{1}{2}\right)-2\ln(2)Li_2\left(\frac{1}{2}\right)-ln^3(2)+\frac{ln^3(2)}{2}=2\zeta(3)-$$

$$2\left\{\frac{ln^3(2)}{6}-\frac{\pi^2\ln(2)}{12}+\frac{7\zeta(3)}{8}\right\}-2\ln(2)\left\{\frac{\pi^2}{12}-\frac{ln^2(2)}{2}\right\}-\frac{2ln^3(2)}{3}=\frac{\zeta(3)}{4}$$

Hence: $\Omega=\frac{1}{2}\left\{-2\zeta(3)+\frac{\zeta(3)}{4}\right\}=-\frac{7}{8}\zeta(3)$

Note: ζ(3)=Apery's constant