ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \frac{x\ln^2(x)}{x^2+x+1}dx$$

Proposed by Shirvan Tahirov-AZerbaijan

Solution 1 by Pham Duc Nam-Vietnam

We have: $\sum_{n=1}^{\infty}\sin(n\theta)\,x^n = \frac{x\sin(\theta)}{x^2-2x\cos(\theta)+1}$

=>Let: $\theta=\frac{2\pi}{3}$ => $\frac{x}{x^2+x+1} = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\sin(\frac{2\pi n}{3})x^n$

=>$\Omega=\frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\sin(\frac{2\pi n}{3})\int_0^1 x^n\ln^2(x)dx = \frac{4}{\sqrt{3}}\sum_{n=1}^{\infty}\sin(\frac{2\pi n}{3})\frac{1}{(n+1)^3}$

And we also have:

$$\text{Sin}(\frac{2\pi n}{3})=\begin{cases} 0, if\ n = 0\ (mod\ 3) \\ \frac{\sqrt{3}}{2}, if\ n = 1(mod 3) \\ -\frac{\sqrt{3}}{2}, if\ n = 2(mod 3)\end{cases}$$

$$>=\Omega=2\sum_{n=0}^{\infty}\frac{1}{(3k+2)^3} - 2\sum_{n=0}^{\infty}\frac{1}{(3k+3)^3} = 2\left(\frac{13}{27}\zeta(3) - \frac{2}{81\sqrt{3}}\pi^3\right) - \frac{2}{27}\zeta(3) = \frac{8}{9}\zeta(3) - \frac{4}{81\sqrt{3}}$$

Note: $\psi^{(2)}\left(\frac{2}{3}\right) = \frac{\pi}{2}\frac{d^2}{dz^2}\cot(\pi z)]_{z=\frac{1}{3}} - (2!)9\left(\zeta(3) + Cl_3\left(\frac{2\pi}{3}\right)\right)$

And : $Cl_{2k+1}\left(\frac{2\pi}{3}\right) = \frac{1}{2}(1-3^{-2k})\zeta(2k+1)$

$$=>\psi^{(2)}\left(\frac{2}{3}\right) = \frac{4}{3\sqrt{3}}\pi^3 - 18\left(\zeta(3) + \frac{8}{18}\zeta(3)\right) = \frac{4}{3\sqrt{3}}\pi^3 - 26\zeta(3)$$

$$\text{But:}\sum_{k=0}^{\infty}\frac{1}{(3k+2)^3} = -\frac{1}{54}\left(\frac{4}{3\sqrt{3}}\pi^3 - 26\zeta(3)\right) = \frac{13}{27}\zeta(3) - \frac{2}{81\sqrt{3}}\pi^3$$

Solution 2 by Bui Hong Suc-Vietnam

$$\int_0^1 x^n\ln^2(x)dx = x^{n+1}[\frac{\ln^2(x)}{n+1} - \frac{2\ln(x)}{(n+1)^2} + \frac{2}{(n+1)^3}]$$

$$\zeta(3,\tfrac{2}{3}) = \sum_{n=0}^{\infty}\frac{1}{(n+\frac{2}{3})^3} = 13\zeta(3) - \frac{2\pi^3}{3\sqrt{3}}$$

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$$\Omega_{n,k}=\int_0^1 \frac{x^k ln^2(x)}{\sum_{i=0}^{n-1} x^i}dx=\int_0^1 \frac{x^k(1-x)ln^2(x)dx}{(1-x)\sum_{i=0}^{n-1} x^i}=\int_0^1 \frac{(x^k-x^{k+1})ln^2(x)}{1-x^n}dx=$$

$$\sum_{j=0}^{\infty}\int_0^1 (x^k-x^{k+1})x^{nj}ln^2(x)dx=\sum_{j=0}^{\infty}\left\{\int_0^1 x^{nj+k}ln^2(x)dx-\int_0^1 x^{nj+k+1}ln^2(x)dx\right\}=$$

$$\sum_{j=0}^{\infty}\left\{\begin{matrix} x^{nj+k+1}\left[\frac{ln^2(x)}{nj+k+1}-\frac{2\ln(x)}{(nj+k+1)^2}+\frac{2}{(nj+k+1)^3}\right] \\ -x^{nj+k+2}\left[\frac{ln^2(x)}{nj+k+2}-\frac{2\ln(x)}{(nj+k+2)^2}+\frac{2}{(nj+k+2)^3}\right]\end{matrix}\right\}\begin{matrix}1\\0\end{matrix}$$

$$=\sum_{j=0}^{\infty}\left\{\frac{2}{(nj+k+1)^3}-\frac{2}{(nj+k+2)^3}\right\}=\frac{2}{n^3}\sum_{j=0}^{\infty}\left\{\frac{1}{(j+\frac{k+1}{n})^3}-\frac{1}{(j+\frac{k+2}{n})^3}\right\}=$$

$$\frac{2}{n^3}\left(\zeta\left(3,\frac{k+1}{n}\right)-\zeta\left(3,\frac{k+2}{n}\right)\right)$$

As : n=3,k=1 $\Omega=\int_0^1 \frac{x ln^2(x)}{x^2+x+1}dx=\frac{2}{3^3}\left(\zeta\left(3,\frac{2}{3}\right)-\zeta\left(3,\frac{3}{3}\right)\right)=$

$$\frac{2}{27}\left(13\zeta(3)-\frac{2\pi^3}{3\sqrt{3}}-\zeta(3)\right)=\frac{8}{9}\zeta(3)-\frac{4\pi^3}{81\sqrt{3}}$$

Note: ζ(3) -> Apery's constant