ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^1\left(x\ln\left(\arccos^2(1-x^2)\right)+\frac{\ln^2(1-x)}{x}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

We have $\overbrace{\int_0^1 x\ln\left(\arccos^2(1-x^2)\right)dx}^{x^2\to x}+\overbrace{\int_0^1\frac{\ln^2(1-x)}{x}dx}^{1-x\to}=$

$\overbrace{\int_0^1 \ln\cos^{-1}(1-x)dx}^{1-x\to}+\overbrace{\int_0^1\frac{\ln^2(x)}{1-x}}^{|x|<1}=\overbrace{\int_0^1 \ln\cos^{-1}(x)dx}^{\cos^{-1}(x)\to x}+\underbrace{\overbrace{\sum_{n\in N}\int_0^1 x^{n-1}\ln^2(x)dx}^{\text{Note section }(1)}}_{\text{Note section}(2)}=$

$\int_0^{\frac{\pi}{2}}\sin(x)\ln(x)\,dx+2\sum_{n\in N}\frac{1}{n^3}\overset{\frown}{=>}\ln(\frac{\pi}{2})-Cin(\frac{\pi}{2})+2\zeta(3)\quad\overset{\text{Note section }(3)}{\overset{\frown}{=>}}\quad=$

$$Ci(\tfrac{\pi}{2})-\gamma-\ln\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi}{2}\right)+2\zeta(3)$$

$$\int_0^1\left(x\ln\left(\arccos^2(1-x^2)\right)+\frac{\ln^2(1-x)}{x}\right)dx=2\zeta(3)+Ci(\tfrac{\pi}{2})-\gamma$$

Note Section:

1) $\int_0^1 x^m\ln^n(x)dx=\frac{(-1)^n n!}{(m+1)^{n+1}}$

2) $\int_0^x\frac{1-\cos(t)}{t}dt=Cin(x)$

3) $Ci(x)=\ln(x)-Cin(x)+\gamma$

Solution 2 by Amin Hajiyev-Azerbaijan

$$\int_0^1\left(x\ln\left(\arccos^2(1-x^2)\right)+\frac{\ln^2(1-x)}{x}\right)dx=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1 x\ln\left(\arccos^2(1-x^2)\right)dx\overset{1-x^2\to x}{\triangleq}$$

$$\Omega_1=\int_0^1\log(\arccos(x))\,dx\to\left\{\arccos(x)=t;\ \frac{dt}{dx}=\frac{1}{\sin(t)}\right\}=$$

$$\Omega_1=\int_0^{\frac{\pi}{2}}\sin(t)\ln(t)\,dt\overset{IBP}{\triangleq}-\Big|_0^{\frac{\pi}{2}}\cos(t)\ln(t)+\int_0^{\frac{\pi}{2}}\frac{\cos(t)}{t}dt=$$

$$\frac{\pi}{2}\ln(0)+\int_0^{\frac{\pi}{2}}\frac{\cos(t)}{t}dt=-\int_0^{\frac{\pi}{2}}\frac{1}{t}dt+\int_0^{\frac{\pi}{2}}\frac{\cos(t)}{t}dt+\ln\left(\frac{\pi}{2}\right)=$$

$$\ln\left(\frac{\pi}{2}\right)+\int_0^{\frac{\pi}{2}}\frac{\cos(t)-1}{t}dt=Ci\left(\frac{\pi}{2}\right)-\gamma$$

$$\Omega_2=\int_0^1\frac{ln^2(1-x)}{x}dx\overset{1-x\to x}{\cong}\int_0^1\frac{ln^2(x)}{1-x}dx=\sum_{n=0}^{\infty}\int_0^1 x^n ln^2(x)dx\overset{IBP}{\cong}2\zeta(3)$$

$$\Omega=\Omega_1+\Omega_2=2\zeta(3)+Ci\left(\frac{\pi}{2}\right)-\gamma$$

Note section:

Cosine integral $Ci(z)=\ln(z)+\gamma+\int_0^z\frac{\cos(x)-1}{x}dx$

For $a\in Z$**, the following identity holds:** $\int_0^1\frac{ln^a(x)}{1-x}dx=(-1)^a\Gamma(a+1)\zeta(a+1)$