ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^1 \frac{\ln(x^2+1)+\arctan(x)}{x^2+1}\,dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$\Omega = \int_0^1 \frac{\ln(x^2+1)+\arctan(x)}{x^2+1}\,dx \overset{\arctan(x)\to x}{\triangleq} \int_0^{\frac{\pi}{4}} \left[\ln\left(1+tan^2(x)\right)+x\right]dx$$

$$= \left(\frac{x^2}{2}\right)\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln\left(sec^2(x)\right)dx =$$

$$\Omega = \frac{1}{2}\left(\frac{\pi^2}{16}-0\right) - 2\left[-\int_0^{\frac{\pi}{4}} \ln(2)\,dx - \sum_{n\in N}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)\,dx\right] =$$

$$\frac{\pi^2}{32} + 2\left[\frac{\pi}{4}\ln(2) + \sum_{n\in N}\frac{(-1)^n}{n}\left(\frac{\sin(2nx)}{2n}\right)\Big|_0^{\frac{\pi}{4}}\right] =$$

$$\frac{\pi^2}{32} + \frac{\pi}{2}\ln(2) + \sum_{n\in N}\frac{(-1)^n}{n}\sin\left(\frac{\pi n}{2}\right) = \frac{\pi^2}{32} + \frac{\pi}{2}\ln(2) - \sum_{n\in N}\frac{(-1)^n}{(2n+1)^2} =$$

$$= \frac{\pi^2}{32} + \frac{16\pi}{32}\ln(2) - G$$

Note section:

$$\begin{cases} \ln(\cos(x)) = -\ln(2) - \sum_{n\in N}\frac{(-1)^n}{n}\cos(2nx) \\ sin^2(x)+cos^2(x) = 1 \to 1+tan^2(x) = sec^2(x) \\ Dirichlet\ Beta\ Function: \beta(2) \\ \beta(2) = \sum_{n\in N}\frac{(-1)^n}{(2n+1)^2} = G\ (Catalan's constant) \end{cases}$$