ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^\infty\int_0^1 \frac{\log(1+y)\log^2(1+x^2)}{x(1+y^2)}\,dx\,dy=\frac{G\zeta(3)}{8}+\frac{\pi}{32}\log(2)$$

where, $\zeta(3)$ is the *Apery's constant.*

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^\infty\int_0^1 \frac{\log(1+y)\log^2(1+x^2)}{x(1+y^2)}\,dx\,dy=\int_0^1\frac{\log^2(1+x^2)}{x}dx*\int_0^\infty\frac{\log(1+y)}{1+y^2}dy=I_1*I_2$$

$$I_1=\int_0^1\frac{\log^2(1+x^2)}{x}dx\Bigg]_{x^2=m}=\frac{1}{2}\int_0^1\frac{\log^2(1+m)}{m}dm=\frac{1}{8}\zeta(3)$$

$$I_2=\int_0^\infty\frac{\log(1+y)}{1+y^2}dy=\int_0^1\frac{\log(1+y)}{1+y^2}dy+\int_1^\infty\frac{\log(1+y)}{1+y^2}dy$$

$$=\int_0^1\frac{\log(1+y)}{1+y^2}dy+\left\{\int_0^1\frac{\log(1+y)}{1+y^2}dy-\int_0^1\frac{\log(y)}{1+y^2}dy\right\}$$

$$=2\int_0^1\frac{\log(1+y)}{1+y^2}dy-\int_0^1\frac{\log(y)}{1+y^2}dy=\frac{\pi}{4}\log(2)+G$$

$$I=I_1*I_2=\frac{1}{8}\zeta(3)\left\{\frac{\pi}{4}\log(2)+G\right\}=\frac{G\zeta(3)}{8}+\frac{\pi}{32}\zeta(3)\log(2)$$

Note: $\int_0^1\frac{\log^2(1+m)}{m}dm=\frac{1}{4}\zeta(3)$ and $\int_0^1\frac{\log(1+y)}{1+y^2}dy=\frac{\pi}{8}\log(2)$