ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_1^{\infty}\frac{x\ln(x)(1-\ln(x))^2}{(1+x^2)(1+x)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_1^{\infty}\frac{x\ln(x)(1-\ln(x))^2}{(1+x)^2(1+x^2)}dx=\frac{1}{2}\underbrace{\int_1^{\infty}\frac{\ln(x)(1-\ln(x))^2}{(1+x^2)}dx}_{\Omega_1}-\frac{1}{2}\underbrace{\int_1^{\infty}\frac{\ln(x)(1-\ln(x))^2}{(1+x)^2}dx}_{\Omega_2}$$

$$\Omega_1=\int_1^{\infty}\frac{\ln(x)-2ln^2(2)+ln^3(x)}{1+x^2}dx=-\int_0^1\frac{\ln(x)+2ln^2(x)+ln^3(x)}{1+x^2}dx=$$

$$-\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}(\ln(x)+2ln^2(x)+ln^3(x))dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}+$$

$$6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}=G-\frac{\pi^3}{8}+\frac{6}{i}\sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)^4}\underbrace{\overset{\{\sum_{n=0}^{\infty}a_{2n+1}=\frac{1}{2}\sum_{n=1}^{\infty}a_n-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_{2n+1}\}}{=}}_{converting\ the\ summand}$$

$$G-\frac{\pi^3}{8}+\frac{3}{i}\sum_{n=1}^{\infty}\frac{i^n}{n^4}-\frac{3}{i}\sum_{n=1}^{\infty}\frac{(-1)^n i^n}{n^4}=G-\frac{\pi^3}{8}-3iLi_4(i)+3iLi_4(-i)$$

$$\Omega_2=\int_1^{\infty}\frac{\ln(x)(1-\ln(x))^2}{(1+x)^2}dx$$

$$=-\int_0^1\frac{\ln(x)(1+\ln(x))^2}{(1+x)^2}dx=-\int_0^1\frac{\ln(x)+2ln^2(x)+ln^3(x)}{(1+x)^2}dx=$$

$$\Omega_2=\sum_{n=1}^{\infty}(-1)^n n\int_0^1 x^{n-1}\left(\ln(x)+2ln^2(x)+ln^3(x)\right)dx\ \overset{\left\{\int_0^1 x^{n-1}ln^k(x)dx=\frac{(-1)^k k!}{n^{k+1}}\right\}}{\underbrace{=}}\ =$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-6\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^4}=\ln(2)-4\eta(2)+6\eta(3)\underbrace{\overset{\{\eta(z)=(1-2^{1-z})\zeta(z)\}}{=}}_{Dirichlet\ eta\ function}$$

$$\ln(2)-2\zeta(2)+\frac{9}{2}\zeta(3)=\ln(2)+\frac{9}{2}\zeta(3)-\frac{\pi^2}{3}$$

$$\Omega=\frac{1}{2}(\Omega_1-\Omega_2)=\frac{G}{2}+\frac{3}{2}iLi_4(-i)-\frac{3}{2}iLi_4(i)+\frac{\pi^2}{6}-\frac{\ln(2)}{2}-\frac{9}{4}\zeta(3)-\frac{\pi^3}{16}$$

Solution 2 by Gbenga Ajeigbe-Nigeria

$$I=\int_1^{\infty}\frac{xln(x)(1-\ln(x))^2}{(1+x^2)(1+x)^2}dx=\int_0^1\frac{xln(\frac{1}{x})(1-\ln\left(\frac{1}{x}\right))^2}{(x^2+1)(x+1)^2}dx=\int_0^1\frac{xln(\frac{1}{x})(1-2\ln\left(\frac{1}{x}\right)+ln^2\left(\frac{1}{x}\right))}{(x^2+1)(x+1)^2}dx=$$

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$$I=\int_0^1 \frac{xln(\frac{1}{x})}{(x^2+1)(x+1)^2}dx-2\int_0^1 \frac{xln^2(\frac{1}{x})}{(x^2+1)(x+1)^2}dx+\int_0^1 \frac{xln^3(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=A-2B+C$$

$$A=\int_0^1 \frac{xln(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=-\int_0^1 \frac{xln(x)}{(x^2+1)(x+1)^2}dx=\frac{1}{2}\int_0^1 \frac{x^2ln^2(x)}{x^2+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{x+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{(x+1)^2}dx$$

$$=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+3)^2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2+n)^2}+\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{(2+n)^2}=-\frac{1}{2}(1-G)+\frac{1}{2}\left(1-\frac{\pi^2}{12}\right)+\frac{1}{2}(\frac{\pi^2}{12}-\ln(2))=\frac{G}{2}-\frac{\ln(2)}{2}$$

$$B=-2\int_0^1 \frac{xln^2(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=-2\int_0^1 \frac{xln^2(x)}{(x^2+1)(x+1)^2}dx=\int_0^1 \frac{x^2\ln(x)}{x^2+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{x+1}dx-\frac{1}{2}\int_0^1 \frac{xln(x)}{(x+1)^2}dx=2!\sum_{n=0}^{\infty}\frac{(-1)^n}{(3+2n)^3}-2!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2+n)^3}-2!\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{(2+n)^3}=$$

$$2(\frac{1}{32}(32-\pi^3))-2(\frac{1}{4}(4-3\zeta(3)))-2(\frac{1}{12}(9\zeta(3)-\pi^2))=\frac{\pi^2}{6}-\frac{\pi^3}{16}$$

$$C=\int_0^1 \frac{xln^3(\frac{1}{x})}{(x^2+1)(x+1)^2}dx=-\int_0^1 \frac{xln^3(x)}{(x^2+1)(x+1)^2}dx=\frac{1}{2}\int_0^1 \frac{x^2ln^3(x)}{x^2+1}dx-$$

$$\frac{1}{2}\int_0^1 \frac{xln^3(x)}{x+1}dx-\frac{1}{2}\int_0^1 \frac{xln^3(x)}{(x+1)^2}dx=-\frac{3!}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(3+2n)^4}+\frac{3!}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2+n)^4}+$$

$$\frac{3!}{2}\sum_{n=0}^{\infty}\frac{\binom{-2}{n}}{(2+n)^4}=\frac{3}{2}iLi_4(-i)-\frac{3}{2}iLi_4(i)-\frac{9\zeta(3)}{4}$$

$$I=A-2B+C$$

$$I=\frac{G}{2}+\frac{3}{2}iLi_4(-i)-\frac{3}{2}iLi_4(i)-\frac{9}{4}\zeta(3)-\frac{\ln(2)}{2}+\frac{\pi^2}{6}-\frac{\pi^3}{16}$$

Note : $\begin{cases} G\to Catalan's\ constant \\ \zeta(3)\to Apery's\ constant \end{cases}$