ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n\psi^{(0)}(n)}{n^2}$$

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$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n(H_{n-\gamma}-1)}{n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^2}\left(H_n-\frac{1}{n}-\gamma\right)=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_nH_n}{n^2}-$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^3}-\gamma\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n}{n^2}=\Omega_1-\Omega_2-\gamma\Omega_3$$

$$\Omega_1=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_nH_n}{n^2}\overset{\left(\overline{H}_n=\ln(2)-\int_0^1\frac{(-x)^n}{1+x}dx\right)}{\cong}\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}$$
$$-\sum_{n=1}^{\infty}\frac{H_n}{n^2}\int_0^1\frac{x^n}{1+x}dx$$

$$=\ln(2)\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}-\int_0^1\frac{Li_3(x)}{1+x}dx+\int_0^1\frac{Li_3(1-x)}{1+x}dx-\int_0^1\frac{\ln(1-x)Li_2(1-x)}{1+x}dx-$$
$$\frac{1}{2}\int_0^1\frac{\ln(x)ln^2(1+x)}{1+x}dx-\zeta(3)\int_0^1\frac{1}{1+x}dx=J_1\ln(2)-J_2+J_3-J_4-J_5-J_6\zeta(3)$$

$$J_1=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}$$
$$=-\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1x^{n-1}\ln(x)\,dx$$
$$=-\int_0^1\frac{\ln(x)}{x}\left(\sum_{n=1}^{\infty}(-1)^nH_nx^n\right)dx$$
$$=\int_0^1\frac{\ln(x)\ln(1+x)}{x(1+x)}dx=\int_0^1\frac{\ln(x)\ln(1+x)}{x}dx-\underbrace{\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}dx}_{IBP}=$$
$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1x^{n-1}\ln(x)\,dx-$$
$$\left[\frac{1}{2}\ln(x)ln^2(1+x)\right]_0^1+\frac{1}{2}\int_0^1\frac{ln^2(1+x)}{x}dx=-\frac{3\zeta(3)}{4}+\frac{1}{2}\int_0^1\frac{ln^2(1+x)}{x}dx=-\frac{5\zeta(3)}{8}$$

$$J_2=\int_0^1\frac{Li_3(x)}{1+x}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1x^nLi_3(x)dx=-\sum_{n=1}^{\infty}(-1)^n\int_0^1x^{n-1}Li_3(x)dx$$
$$\overset{\left\{\int_0^1x^{n-1}Li_3(x)=\frac{\zeta(3)}{n}-\frac{\zeta(2)}{n^2}+\frac{H_n}{n^3}\right\}}{\cong}-\sum_{n=1}^{\infty}(-1)^n\left(\frac{\zeta(3)}{n}-\frac{\zeta(2)}{n^2}+\frac{H_n}{n^3}\right)=$$
$$-\zeta(3)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}=\zeta(3)\ln(2)-\frac{1}{2}\zeta^2(2)-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}$$

$$=\zeta(3)\ln(2)-\frac{\pi^4}{72}-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^{n-1}ln^2(x)dx=\zeta(3)\ln(2)-\frac{\pi^4}{72}+\frac{1}{2}\underbrace{\int_0^1\frac{\ln(1+x)ln^2(x)}{x(1+x)}dx}_{subtitation\to x=\frac{t}{1-t}}$$

$$=\zeta(3)\ln(2)-\frac{\pi^4}{72}-\frac{1}{2}\int_0^{\frac{1}{2}}\frac{ln^2\left(\frac{t}{1-t}\right)\ln(1-t)}{t}dt=\zeta(3)\ln(2)-\frac{\pi^4}{72}-\frac{1}{2}(\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-t)ln^2(t)}{t}dt+\int_0^{\frac{1}{2}}\frac{ln^3(1-t)}{t}dt}_{L}$$

$$-2\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}}_{P}\quad L=\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-t)ln^2(t)}{t}dt}_{IBP}+\int_0^{\frac{1}{2}}\frac{ln^3(t)}{t}dt=\left[\frac{1}{3}\ln(1-t)ln^3(t)\right]_0^{\frac{1}{2}}+\frac{1}{3}\int_0^{\frac{1}{2}}\frac{ln^3(t)}{1-t}dt+$$

$$\underbrace{\int_0^{\frac{1}{2}}\frac{ln^3(1-t)}{t}dt}_{1-t\to t}=\frac{1}{3}ln^4(2)+\frac{1}{3}\int_0^{\frac{1}{2}}\frac{ln^3(t)}{1-t}dt-\int_0^{\frac{1}{2}}\frac{ln^3(t)}{1-t}dt+\int_0^1\frac{ln^3(t)}{1-t}dt=$$

$$4Li_4\left(\frac{1}{2}\right)+\frac{7}{2}\zeta(3)ln(2)+\frac{2}{3}ln^4(2)-\frac{\pi^2}{6}ln^4(2)-\frac{\pi^4}{15}$$

$$P=\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}dt\overset{IBP}{\cong}\frac{1}{2}ln^4(2)+\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-t)ln^2(t)}{1-t}dt}_{1-t\to t}=\frac{1}{2}ln^4(2)-\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}dt$$

$$=\frac{1}{2}ln^4(2)-\int_0^{\frac{1}{2}}\frac{\ln(t)ln^2(1-t)}{t}dt+\int_0^1\frac{ln^2(1-t)\ln(t)}{t}dt=\frac{1}{2}ln^4(2)+\underbrace{\int_0^1\frac{\ln(t)ln^2(1-t)}{t}dt}_{1-t=>t}-P$$

$$2P=\frac{1}{2}ln^4(2)+2\zeta(4)-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{1}{2}ln^4(2)+2\zeta(4)-\frac{5}{2}\zeta(4)\quad P=\frac{1}{4}ln^4(2)-\frac{\pi^4}{360}$$

$$J_2=\zeta(3)\ln(2)-\frac{\pi^4}{72}-\frac{1}{2}(L-2P)$$

$$=\frac{\pi^2}{12}ln^2(2)-\frac{1}{12}ln^4(2)-2Li_4\left(\frac{1}{2}\right)+\frac{\pi^4}{60}-\frac{3}{4}\zeta(3)\ln(2)$$

$$J_3=\int_0^1\frac{Li_3(1-x)}{1+x}dx\overset{1-x=>x}{\cong}\int_0^1\frac{Li_3(x)}{2-x}dx=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 x^{n-1}Li_3(x)dx=$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{\zeta(3)}{n}-\frac{\zeta(2)}{n^2}+\frac{H_n}{n^3}\right)=\zeta(3)Li_1\left(\frac{1}{2}\right)-\zeta(2)Li_2\left(\frac{1}{2}\right)+\sum_{n=1}^{\infty}\frac{H_n}{2^n n^3}$$

$$\left\{\int_0^{\frac{1}{2}}x^{n-1}ln^2(x)dx=\frac{ln^2(2)}{n2^n}+\frac{2\ln(2)}{n^2 2^n}+\frac{2}{n^3 2^n}\right\}$$

$$\cong\quad\zeta(3)\ln(2)-\frac{\pi^4}{72}+\frac{\pi^2}{12}ln^2(2)-\frac{1}{6}ln^4(2)-\frac{1}{2}ln^2(2)\sum_{n=1}^{\infty}\frac{H_n}{n2^n}-$$

$$\ln(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2 2^n}-\frac{1}{6}\underbrace{\int_0^{\frac{1}{2}}\frac{ln^3(x)}{1-x}dx}-\frac{1}{2}\underbrace{\int_0^{\frac{1}{2}}\frac{\ln(1-x)ln^2(x)}{1-x}dx}_{-\frac{1}{4}ln^4(2)-\frac{1}{4}\zeta(4)}=$$

$$\left\{\int_0^{\frac{1}{2}}\frac{ln^n(x)}{1-x}dx=\sum_{n=0}^{a}(-1)^n n!\binom{a}{n}((-\ln(2))^{a-n}Li_{n+1}\left(\frac{1}{2}\right)\right\}$$

$$\zeta(3)\ln(2)-\frac{\pi^4}{72}+\frac{\pi^2}{12}ln^2(2)+Li_4\left(\frac{1}{2}\right)+\frac{\pi^4}{720}-\frac{1}{8}\ln(2)\zeta(3)+\frac{1}{24}ln^4(2)=$$

$$\frac{7}{8}\zeta(3)\ln(2)-\frac{\pi^4}{80}+\frac{\pi^2}{12}ln^2(2)+Li_4\left(\frac{1}{2}\right)+\frac{ln^4(2)}{24}$$

$$J_4=\int_0^1\frac{\ln(1-x)Li_2(1-x)}{1+x}dx\overset{1-x\to x}{\cong}\int_0^1\frac{\ln(x)Li_2(x)}{2-x}dx$$

$$=\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 x^{n-1}\ln(x)\,Li_2(x)dx=$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\partial}{\partial n}\int_0^1 x^{n-1}Li_2(x)dx\overset{\left\{\int_0^1 x^{n-1}Li_2(x)dx=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\right\}}{\cong}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\partial}{\partial n}\left(\frac{\zeta(2)}{n}-\frac{H_n}{n^2}\right)=$$

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{2H_n}{n^3}+\frac{H^{(2)}{}_n}{n^2}-\frac{2\zeta(2)}{n^2}\right)=3Li_4\left(\frac{1}{2}\right)-\frac{7\pi^4}{288}+\frac{ln^4(2)}{8}+\frac{\pi^2}{8}ln^2(2)$$

$$J_5=\frac{1}{2}\int_0^1\frac{\ln(x)ln^2(1-x)}{1+x}dx\overset{1-x\to x}{\cong}\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(x)}{2-x}dx$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1\frac{\partial^2}{\partial n^2}x^{n-1}\ln(1-x)\,dx$$

$$\overset{\left\{\int_0^1 x^{n-1}\ln(1-x)dx=-\frac{H_n}{n}\right\}}{\cong}\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\partial^2}{\partial n^2}\left(-\frac{H_n}{n}\right)$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}\left(\frac{2\zeta(3)}{n}+\frac{2\zeta(2)}{n^2}-\frac{2H_n}{n^3}-\frac{2H^{(2)}_n}{n^2}-\frac{2H^{(3)}_n}{n}\right)$$

$$2J_5=2\ln(2)\zeta(3)+2\zeta(2)Li_2\left(\tfrac{1}{2}\right)-2\sum_{n=1}^{\infty}\frac{H_n}{2^nn^3}-2\sum_{n=1}^{\infty}\frac{H^{(2)}_n}{2^nn^2}-2\sum_{n=1}^{\infty}\frac{H^{(3)}_n}{2^nn}$$

$$=2\ln(2)\zeta(3)+\frac{\pi^4}{36}-\frac{\pi^2}{6}ln^2(2)-2\sum_{n=1}^{\infty}\frac{H_n}{n^32^n}-6Li_4\left(\frac{1}{2}\right)-Li_2^2\left(\frac{1}{2}\right)+4\sum_{n=1}^{\infty}\frac{H_n}{n^32^n}-\int_0^1\frac{\ln(1+x)ln^2(1-x)}{x}dx$$

$$=2\ln(2)\zeta(3)+\frac{\pi^4}{36}-\frac{\pi^2}{6}ln^2(2)-2\sum_{n=1}^{\infty}\frac{H_n}{n^32^n}-6Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{144}-\frac{ln^4(2)}{4}+\frac{\pi^2}{12}ln^2(2)-\underbrace{\int_0^1\frac{\ln(1+x)ln^2(1-x)}{x}dx}_{T}$$

$$T=\int_0^1\frac{\ln(1+x)ln^2(1-x)}{x}dx=\frac{1}{6}\underbrace{\int_0^1\frac{ln^3(1-x^2)}{x}dx}_{1-x^2\to x}-\frac{1}{6}\underbrace{\int_0^1\frac{ln^3\left(\frac{1-x}{1+x}\right)}{x}dx}_{x\to\frac{1-x}{1+x}}-\frac{1}{3}\underbrace{\int_0^1\frac{ln^3(1+x)}{x}dx}$$

$$=2Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{144}+\frac{7}{4}\ln(2)\zeta(3)-\frac{\pi^2}{12}ln^2(2)+\frac{1}{12}ln^4(2)$$

$$2J_5=-6Li_4\left(\frac{1}{2}\right)+\frac{11\pi^4}{360}-\frac{1}{4}ln^4(2)\to J_5=-3Li_4\left(\frac{1}{2}\right)+\frac{11\pi^4}{720}-\frac{1}{8}ln^4(2)$$

$$J_6=\zeta(3)\int_0^1\frac{1}{1+X}dx=\zeta(3)\ln(2)$$

$$\Omega_1=\ln(2)J_1-J_2+J_3-J_4-J_5-\zeta(3)J_6=3Li_4\left(\frac{1}{2}\right)-\frac{29\pi^4}{1440}-\frac{\pi^2}{8}ln^2(2)+\frac{1}{8}ln^4(2)$$

$$\Omega_2=\sum_{n=1}^{\infty}\frac{(-1)^n\overrightarrow{H_n}}{n^3}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\overleftrightarrow{H_n}\int_0^1 x^{n-1}ln^2(x)dx=\frac{1}{2}\int_0^1\frac{\ln(1-x)ln^2(x)}{x(1+x)}dx$$

$$=\frac{1}{2}\Big(\underbrace{\int_0^1\frac{\ln(1-x)ln^2(x)}{x}dx}_{IBP}-\int_0^1\frac{\ln(1-x)ln^2(x)}{1+x}dx\Big)=\frac{1}{6}\underbrace{\int_0^1\frac{ln^3(x)}{1-x}dx}_{\left\{\int_0^1\frac{ln^a(x)}{1-x}dx=(-1)^a a!\zeta(a+1)\right\}}-$$

$$-\frac{1}{2}\underbrace{\int_0^1\frac{\ln(1-x)ln^2(x)}{1+x}}_{-4Li_4\left(\frac{1}{2}\right)+\zeta(4)+ln^2(2)\zeta(2)-\frac{1}{6}ln^4(2)}=2Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{60}-\frac{\pi^2}{12}ln^2(2)+\frac{1}{12}ln^4(2)$$

$$\Omega_3=\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H_n}}{n^2}=-\sum_{n=1}^{\infty}(-1)^n\overline{H_n}\int_0^1 x^{n-1}\ln(x)\,dx=-\int_0^1\frac{\ln(x)\ln(1-x)}{x(1+x)}dx$$

$$=\underbrace{\int_0^1\frac{\ln(x)\ln(1-x)}{1+x}dx}_{\frac{13}{8}\zeta(3)-\frac{3}{2}\ln(2)\zeta(2)}-\underbrace{\int_0^1\frac{\ln(x)\ln(1-x)}{x}dx}_{\zeta(3)}=\frac{13}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)-\zeta(3)=\frac{5}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)$$

$$\Omega=\Omega_1-\Omega_2-\gamma\Omega_3=3Li_4\left(\frac{1}{2}\right)-\frac{29\pi^4}{1440}-\frac{\pi^2}{8}ln^2(2)+\frac{1}{8}ln^4(2)-2Li_4\left(\frac{1}{2}\right)+\frac{\pi^4}{60}+\frac{\pi^2}{12}ln^2(2)-\frac{1}{12}ln^4(2)-\gamma\left(\frac{5}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)\right)$$

Answer:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overline{H}_n\psi^{(0)}(n)}{n^2}=Li_4\left(\frac{1}{2}\right)-\frac{\pi^4}{2880}+\frac{1}{24}ln^4(2)-\frac{\pi^2ln^2(2)}{24}-\gamma\left(\frac{5}{8}\zeta(3)-\frac{\pi^4}{4}\ln(2)\right)$$

Note section :

Harmonic numbers: $\sum_{k=1}^{n}\frac{1}{k}=H_n\,;\sum_{k=1}^{n}\frac{1}{k^m}=H_n^{(m)}$

$$\frac{\partial}{\partial n}H_n=\zeta(2)-H_n^{(2)};\ \frac{\partial^2}{\partial n^2}H_n=2H_n^{(3)}-2\zeta(3);\ \sum_{n=1}^{\infty}(\pm x)^nH_n=-\frac{\ln(1\pm x)}{1\pm x}$$

$$H_n=\psi^{(0)}(n+1)+\gamma$$

Skew harmonic numbers:

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}=\ln(2)-\int_0^1\frac{(-x)^n}{1+x}=\overline{H_n}$$

$$\sum_{n=1}^{\infty}(-1)^n\overline{H_n}x^n=\frac{\ln(1-x)}{1+x}$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n2^n}=\frac{\zeta(2)}{2};\ \sum_{n=1}^{\infty}\frac{H_n}{n^22^n}=\zeta(3)-\frac{1}{2}\ln(2)\zeta(2)$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^22^n}+2\sum_{n=1}^{\infty}\frac{H_n}{n^32^n}=3Li_4\left(\frac{1}{2}\right)+\frac{1}{2}Li_2^2\left(\frac{1}{2}\right)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^32^n}=Li_4\left(\frac{1}{2}\right)+\ln(2)Li_3\left(\frac{1}{2}\right)-\frac{1}{2}Li_2^2\left(\frac{1}{2}\right)$$

Polylogarithm function:

$$Li_a(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^a}\ ;\ \ Li_a(1)=\zeta(a)\ Li_2\left(\frac{1}{2}\right)=\frac{\pi^{12}}{12}-\frac{ln^2(2)}{2}$$

$$\frac{\partial}{\partial z}Li_a(z)=\frac{Li_{a-1}(z)}{z};\ Li_1(z)=-\ln(1-z)\ ;\ \int_0^1 x^{n-1}Li_2(x)dx=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}$$

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$$\int_0^1 x^{n-1} Li_3(x)dx = \frac{\zeta(3)}{n} - \frac{\zeta(2)}{n^2} + \frac{H_n}{n^3}$$

$$\int_0^{\frac{1}{2}} x^{n-1} ln^2(x)dx = \frac{ln^2(2)}{2^n n} + \frac{2\ln(2)}{2^n n^2} + \frac{2}{2^n n^3}$$