ROMANIAN MATHEMATICAL MAGAZINE

$$If\int_0^1 \frac{arctan^2(x)\log(\frac{x}{1-x^2})}{1+x^2}dx = \frac{7\pi\zeta(a)}{128} - \frac{\pi\zeta(b)\ln(2)}{32}$$

then prove that :

$$\int_0^\infty \frac{(Li_{a+b}(-x) - Li_{a-b}(-x))}{1+x^2}dx = G + \frac{\pi}{4}\ln(2) - \frac{15\pi}{1024}\zeta(5) - 5\beta(6)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\int_0^1 \frac{arctan^2(x)\log(\frac{x}{1-x^2})}{1+x^2}dx \overset{\{\arctan(x)\to x\}}{\widehat{=}} \int_0^{\frac{\pi}{4}} x^2\ln\left(\frac{2\tan(x)}{2(1-tan^2(x))}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} x^2\ln\left(\frac{\tan(2x)}{2}\right)dx =$$

$$\int_0^{\frac{\pi}{4}} x^2\ln(\tan(2x)) - \ln(2))dx = \int_0^{\frac{\pi}{4}} x^2\ln(\tan(2x)) - \ln(2))dx - \ln(2)\int_0^{\frac{\pi}{4}} x^2dx \overset{2x\to x}{\widehat{=}}$$

$$\frac{1}{8}\int_0^{\frac{\pi}{2}} x^2\ln(\tan(x))\,dx - \ln(2)\left.\frac{x^3}{3}\right|_0^{\frac{\pi}{4}} = \frac{1}{8}J - \frac{\pi^3\ln(2)}{192}$$

$$J \overset{Fourier\ series}{\widehat{=}} -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\underbrace{\int_0^{\frac{\pi}{2}} x^2\cos(2x(2n+1))\,dx}_{IBP} = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\left(\frac{\pi^2\sin(2\pi n)}{8(2n+1)^2}\right.$$

$$\left.-\frac{\sin(2\pi n)}{4(2n+1)^3} + \frac{\pi\cos(2\pi n)}{4(2n+1)^2}\right) = \underbrace{\frac{\pi^2}{4}\sum_{n=0}^{\infty}\frac{\sin(2\pi n)}{(2n+1)^3} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{\sin(2\pi n)}{(2n+1)^4}}_{\{\sin(2\pi n)=0,n\in Z\}} + \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\cos(2\pi n)}{(2n+1)^3}$$

$$=\frac{\pi}{4}\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n^3} + \sum_{n=0}^{\infty}\frac{1}{n^3}\right) = \frac{7\pi\zeta(3)}{16}$$

$$\Omega=\frac{1}{8}J - \frac{\pi^3\ln(2)}{192} = \frac{7\pi\zeta(3)}{128} - \frac{\pi\zeta(2)\ln(2)}{32}\ ;\ \Omega = \frac{7\pi\zeta(a)}{128} - \frac{\pi\zeta(b)\ln(2)}{32}\ \{a=3, b=2\}$$

$$I = \int_0^\infty \frac{(Li_{a+b}(-x) - Li_{a-b}(-x))}{1+x^2}dx = \int_0^\infty \frac{Li_5(-x)}{1+x^2}dx - \int_0^\infty \frac{Li_1(-x)}{1+x^2}dx = I_1 - I_2$$

$$I_1 = \int_0^\infty \frac{Li_5(-x)}{1+x^2}dx$$

$$= \int_0^\infty \frac{1}{1+x^2}\left(-\frac{1}{24}\int_0^1 \frac{xln^4(y)}{1+xy}dy\right)dx$$

$$= -\frac{1}{24}\int_0^1 ln^4(y)\left(\int_0^\infty \frac{1}{(1+xy)(1+x^2)}dx\right)dy =$$

$$-\frac{1}{24}\int_0^1 ln^4(y)\left(\frac{\pi}{2}\frac{y}{1+y^2} - \frac{\ln(y)}{1+y^2}\right)dy = -\frac{\pi}{48}\int_0^1 \frac{yln^4(y)}{1+y^2}dy + \frac{1}{24}\int_0^1 \frac{ln^5(y)}{1+y^2}dy = -\frac{\pi}{48}M + \frac{1}{24}N$$

$$M = \int_0^1 \frac{yln^4(y)}{1+y^2}dy$$

$$= \sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{2n+1}ln^4(y)dy \overset{IBP}{\widehat{=}} \frac{24}{32}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^5} = \frac{24}{32}\eta(5) = \frac{45}{64}\zeta(5)$$

$$N=\int_0^1 \frac{ln^5(y)}{1+y^2}dy=\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{2n}ln^5(y)dy \overset{IBP}{=} -120\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^6}=-120\beta(6)$$

$$I_1=-\frac{\pi}{48}M+\frac{1}{24}N=-\frac{15\pi\zeta(5)}{1024}-5\beta(6)$$

Dirichlet beta function : $\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z}$

$$I_2=\int_0^{\infty}\frac{Li_1(-x)}{1+x^2}dx=-\int_0^{\infty}\frac{\ln(1+x)}{1+x^2}dx=-\int_1^{\infty}\frac{\ln(1+x)}{1+x^2}dx-\int_0^1\frac{\ln(1+x)}{1+x^2}dx=$$

$$\int_0^1\frac{\ln(x)}{1+x^2}dx-2\int_0^1\frac{\ln(1+x)}{1+x^2}dx=K-2P$$

$$K=\int_0^1\frac{\ln(x)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(x)\,dx=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-G$$

$$\int_0^1\frac{\ln(1+x)}{1+x^2}dx \overset{\{\arctan(x)\to x\}}{=} \int_0^{\frac{\pi}{4}}\ln(1+\tan(x))\,dx=\underbrace{\int_0^{\frac{\pi}{4}}\ln\left(\sqrt{2}cos\left(\frac{\pi}{4}-x\right)\right)dx}_{\frac{\pi}{4}-x\to x}-$$

$$\int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx=\frac{\ln(2)}{2}\int_0^{\frac{\pi}{4}}dx+\int_0^{\frac{\pi}{4}}\ln(\cos(x))\,dx=\frac{\pi\ln(2)}{8}$$

$$I_2=K-2P=-G-\frac{\pi}{4}\ln(2)$$

Answer:

$$\int_0^{\infty}\frac{Li_5(-x)}{1+x^2}dx-\int_0^{\infty}\frac{Li_1(-x)}{1+x^2}dx=I_1-I_2=G+\frac{\pi}{4}\ln(2)-\frac{15\pi}{1024}\zeta(5)-5\beta(6)$$

Solution 2 by Pham Duc Nam-Vietnam

$$I=\int_0^1\frac{arctan^2(x)\log(\frac{x}{1-x^2})}{1+x^2}dx=\int_0^1\frac{arctan^2(x)\ln(\frac{2x}{1-x^2})-arctan^2(x)\ln(2)}{1+x^2}dx=$$

$$\frac{1}{4}\int_0^1\frac{arctan^2\left(\frac{2x}{1-x^2}\right)\ln(\frac{2x}{1-x^2})}{1+x^2}dx-\frac{1}{3}\ln(2)\,arctan^2(x)\Big|_0^1=I_1-\frac{\pi^3 ln(2)}{192}$$

$$I_1=\frac{1}{4}\int_0^1\frac{arctan^2\left(\frac{2x}{1-x^2}\right)\ln\left(\frac{2x}{1-x^2}\right)}{1+x^2}dx\ \ let: x=\frac{1-t}{1+t}\to dx=-\frac{2}{(1+t)^2}\to$$

$$I_1=-\frac{1}{4}\int_0^1\frac{arctan^2\left(\frac{1-x^2}{2x}\right)\ln\left(\frac{2x}{1-x^2}\right)}{1+x^2}dx\to$$

$$2I_1=\frac{1}{4}\int_0^1\frac{\ln\left(\frac{2x}{1-x^2}\right)\left(arctan^2\left(\frac{2x}{1-x^2}\right)-arctan^2\left(\frac{1-x^2}{2x}\right)\right)}{1+x^2}dx=$$

$$\frac{\pi}{8}\int_0^1\frac{\ln(\frac{2x}{1-x^2})(2\arctan\left(\frac{2x}{1-x^2}\right)-\frac{\pi}{2})}{1+x^2}dx \overset{x\to\tan(x)}{=} \frac{\pi}{8}\int_0^1\ln(\tan(2x))\left(4x-\frac{\pi}{2}\right)dx=$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{4}} xln(\tan(2x))dx - \frac{\pi^2}{16}\int_0^{\frac{\pi}{4}} \ln(\tan(2x))\,dx \overset{x\to 2x}{\cong} \frac{\pi}{8}\int_0^{\frac{\pi}{2}} xln(\tan(x))dx$$

$$-\frac{\pi^2}{32}\int_0^1 xln(\tan(x))dx =$$

$$\frac{\pi}{8}.\frac{7}{8}\zeta(3) - \frac{\pi^2}{32}.0 = \frac{7\pi}{64}\zeta(3) \quad I_1 = \frac{7\pi}{128}\zeta(3) \quad ---->>>>$$

$$I = \frac{7\pi}{128}\zeta(3) - \frac{\pi^3\ln(2)}{192} = \frac{7\pi}{128}\zeta(3) - \frac{\pi\zeta(2)\ln(2)}{32} \quad => a = 3\,, b = 2$$

$$J = \int_0^\infty \frac{Li_5(-x) - Li_1(-x)}{1+x^2}dx = \int_0^\infty \frac{Li_5(-x)}{1+x^2}dx + \int_0^\infty \frac{\ln(1+x)}{1+x^2}dx = J_1 + J_2$$

$$J_1 = \int_0^\infty \frac{Li_5(-x)}{1+x^2}dx = -\frac{1}{24}\int_0^1 ln^4(t)dt\int_0^\infty \frac{x}{(1+x^2)(1+xt)}dx =$$

$$-\frac{1}{24}\int_0^1 \frac{ln^4(t)}{1+t^2}dt\int_0^\infty\left(\frac{x+t}{1+x^2} - \frac{t}{1+xt}\right)dx = -\frac{1}{24}\int_0^1 \frac{ln^4(t)}{1+t^2}dt(tarctan(x) +$$

$$\frac{1}{2}\ln(x^2+1) - \ln(1+xt))\Big|_0^\infty = -\frac{1}{24}\int_0^1 \frac{ln^4(t)}{1+t^2}\left(\frac{\pi}{2}t - \ln(t)\right)dt = -\frac{\pi}{1536}\int_0^1 \frac{ln^4(t^2)}{1+t^2}d(t^2)$$

$$+\frac{1}{24}\int_0^1 \frac{ln^5(t)}{1+t^2}dt = -\frac{\pi}{1536}\int_0^1 \frac{ln^4(t)}{1+t} + \frac{1}{24}\sum_{n=0}^\infty (-1)^n\int_0^1 t^{2n}ln^5(t)dt$$

$$apply: \int_0^1 \frac{ln^n(t)}{1+t}dt = \left(-\frac{1}{2}\right)^n(2^n-1)\zeta(n+1)\Gamma(n+1) n = 4 =>$$

$$\int_0^1 \frac{ln^4(t)}{1+t}dt = \frac{1}{16}.15.\zeta(5).\Gamma(5) = \frac{45}{2}\zeta(5) \quad =>$$

$$J_1 = -\frac{\pi}{1536}.\frac{45}{2}\zeta(5) - \frac{120}{24}.\sum_{n=0}^\infty (-1)^n\frac{1}{(2n+1)^6} = -\frac{15\pi}{1024}\zeta(5) - 5\beta(6)$$

$$J = G + \frac{\pi}{4}\ln(2) - \frac{15\pi}{1024}\zeta(5) - 5\beta(6)$$