ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^{\pi}\left(\sin^4(x)+\cos^2(x)+\frac{x^2}{`1+x^2}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^{\pi}\left(\sin^4(x)+\cos^2(x)+\frac{x^2}{`1+x^2}\right)dx = \Omega_1+\Omega_2+\Omega_3$$

$$\Omega_1 = \int_0^{\pi}\sin^4(x)dx \overset{\cos(x)\to t,\ \frac{dt}{dx}=-\sqrt{1-t^2}}{\triangleq} \int_{-1}^{1}\frac{(1-t^2)^2}{\sqrt{1-t^2}}dt = 2\int_0^1 (1-t^2)^{\frac{3}{2}}dt \overset{t^2\to t}{\triangleq}$$

$$\int_0^1 t^{-\frac{1}{2}}(1-t^2)^{\frac{3}{2}}dt = \beta\left(\frac{1}{2};\frac{5}{2}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{\Gamma(1+\frac{3}{2})\Gamma(1-\frac{3}{2})\sqrt{\pi}}{2\Gamma(-\frac{1}{2})} = -\frac{\frac{3\pi}{2}\csc\left(\frac{3\pi}{2}\right)}{4} = \frac{3\pi}{8}$$

$$\Omega_2 = \int_0^{\pi}\cos^2(x)dx = 2\int_0^1 t^2(1-t^2)^{-\frac{1}{2}}dt = \int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt = \beta\left(\frac{3}{2};\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

$$\Omega_3 = \int_0^{\pi}\frac{x^2}{1+x^2}dx = \int_0^{\pi}dx - \int_0^{\pi}\frac{1}{1+x^2}dx = \pi - \arctan(x)$$

$$\Omega_1+\Omega_2+\Omega_3 = \frac{3\pi}{8}+\frac{\pi}{2}+\pi-\arctan(x) = \frac{15\pi}{8}-\arctan(x)$$

Solution 2 by Cosghun Mammedov-Azerbaijan

$$\Omega = \int_0^{\pi}\left(\sin^4(x)+\cos^2(x)+\frac{x^2}{`1+x^2}\right)dx = \int_0^{\pi}\left((\sin^4(x)-\sin^2(x)+1)+\frac{x^2}{`1+x^2}\right)dx =$$

$$\Omega = \int_0^{\pi}(\sin^4(x)-\sin^2(x)+1)dx + \int_0^{\pi}\frac{x^2}{`1+x^2}dx = \Omega_1+\Omega_2$$

$$\Omega_1 = \int_0^{\pi}\left(\sin^4(x)-\sin^2(x)+\frac{1}{4}+\frac{3}{4}\right)dx = \int_0^{\pi}\left(\left(\sin^2(x)-\frac{1}{2}\right)^2+\frac{3}{4}\right)dx =$$

$$\int_0^{\pi}\left(\left(\frac{1-\cos(2x)}{2}-\frac{1}{2}\right)^2+\frac{3}{4}\right)dx = \int_0^{\pi}\left(\left(-\frac{\cos(2x)}{2}\right)^2+\frac{3}{4}\right)dx = \int_0^{\pi}\left(\frac{\cos^2(2x)}{4}+\frac{3}{4}\right)dx =$$

$$\int_0^{\pi}\left(\frac{\cos(4x)}{8}+\frac{7}{8}\right)dx = \left(\frac{1}{32}\sin(4x)+\frac{7}{8}x\right)_0^{\pi} = \frac{7}{8}\pi$$

$$\Omega_2 = \int_0^{\pi}\frac{x^2}{1+x^2}dx = \int_0^{\pi}\left(1-\frac{1}{1+x^2}\right)dx = (x-\arctan(x))_0^{\pi} = \pi-\arctan(\pi)$$

$$\Omega = \Omega_1+\Omega_2 = \frac{7}{8}\pi+\pi-\arctan(\pi) = \frac{15}{8}\pi-\arctan(\pi)$$