ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{x\cos^2(2x)}{(1+\sin(2x))(1+\cos(2x))}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\int_0^{\frac{\pi}{4}}\frac{x\cos^2(2x)}{(1+\sin(2x))(1+\cos(2x))}dx=\int_0^{\frac{\pi}{4}}\frac{x(1-\sin(2x))}{1+\cos(2x)}\overset{2x\to t}{=}\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{t(1-\sin(t))}{1+\cos(t)}dt=$$

$$\frac{1}{4}\left(\int_0^{\frac{\pi}{2}}\frac{t}{1+\cos(t)}dt-\int_0^{\frac{\pi}{2}}\frac{t\sin(t)}{1+\cos(t)}dt\right)$$

$$\Omega_1=\int_0^{\frac{\pi}{2}}\frac{t}{1+\cos(t)}dt\overset{IBP}{=}\left[t\tan\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\tan\left(\frac{t}{2}\right)dt=\frac{\pi}{2}+2\left[\log\left(\cos\left(\frac{x}{2}\right)\right)\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}-\ln(2)$$

$$\Omega_2=\int_0^{\frac{\pi}{2}}\frac{t\sin(t)}{1+\cos(t)}dt$$

$$=\int_0^{\frac{\pi}{2}}t\tan\left(\frac{t}{2}\right)dt\overset{IBP}{=}-\left[2t\ln\left(\cos\left(\frac{t}{2}\right)\right)\right]_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}\ln\left(\cos\left(\frac{t}{2}\right)\right)dt=$$

$$\frac{\pi}{2}\ln(2)-2\ln(2)\int_0^{\frac{\pi}{2}}dt=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{2}}\cos(nt)\,dt\overset{IBP}{=}\frac{\pi}{2}\ln(2)-\pi\ln(2)-$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\sin(nt)}{n}\right]_0^{\frac{\pi}{2}}=-\frac{\pi}{2}\ln(2)-2\sum_{n=1}^{\infty}\frac{(-1)^n\sin(\frac{\pi n}{2})}{n^2}=2G-\frac{\pi}{2}\ln(2)$$

$$\Omega=\frac{1}{4}(\Omega_1-\Omega_2)=\frac{1}{4}\left(\frac{\pi}{2}-\ln(2)-2G+\frac{\pi}{2}\ln(2)\right)=\frac{\pi}{8}-\frac{\ln(2)}{4}-\frac{G}{2}+\frac{\pi}{8}\ln(2)$$

$$\int_0^{\frac{\pi}{4}}\frac{x\cos^2(2x)}{(1+\sin(2x))(1+\cos(2x))}dx=\frac{\pi}{8}-\frac{\ln(2)}{4}-\frac{G}{2}+\frac{\pi}{8}\ln(2)$$