ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^\infty \frac{x^2+\sin(\pi x)}{1+\exp(\pi x)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\int_0^\infty \frac{x^2+\sin(\pi x)}{1+e^{\pi x}}dx=\int_0^\infty \frac{x^2}{1+e^{\pi x}}dx+\int_0^\infty \frac{\sin(\pi x)}{1+e^{\pi x}}dx=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^\infty \frac{x^2}{1+e^{\pi x}}dx\ \left\{e^{\pi x}=\frac{1}{t}\ \pi x=-\ln(t)\ \ dx=-\frac{1}{\pi t}\right\}$$

$$\Omega_1=-\frac{1}{\pi^3}\int_0^1\frac{ln^2(t)}{1+\frac{1}{t}}\frac{dt}{t}=-\frac{1}{\pi^3}\int_0^1\frac{ln^2(t)}{1+t}dt=-\frac{1}{\pi^3}\sum_{n=0}^\infty(-1)^n\int_0^1 t^n ln^2(t)dt$$

$$\overset{IBP}{=}-\frac{2}{\pi^3}\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^3}=\frac{3\zeta(3)}{2\pi^3}$$

$$\Omega_2=\int_0^\infty\frac{\sin(\pi x)}{1+e^{\pi x}}dx\overset{\{\pi x=t\}}{=}\frac{1}{\pi}\int_0^\infty\frac{\sin(t)}{1+e^t}dt=\frac{1}{\pi}I\ ;\ note\ \{\sinh(it)=i\sin(t)\}$$

$$I(a)=\int_0^\infty\frac{\sinh(at)}{1+e^t}dt\overset{e^t\to t}{=}\frac{1}{2}\underbrace{\int_1^\infty\frac{t^a}{t(t+1)}dt}_{t\to\frac{1}{t}}-\frac{1}{2}\int_1^\infty\frac{t^{-a}}{t(t+1)}dt=$$

$$\frac{1}{2}\int_0^1\frac{t^{-a}}{1+t}dt-\frac{1}{2}\int_1^\infty t^{-a}\left(\frac{1}{t}-\frac{1}{1+t}\right)dt=\frac{1}{2}\left(\underbrace{\int_0^\infty\frac{t^{-a}}{1+t}dt}_{\{\pi\csc(\pi a)\ \Re\{a\}<1\}}-\int_1^\infty t^{-a-1}dt\right)=$$

$$\frac{1}{2}\left(\pi\csc(\pi a)-\frac{1}{a}\right);I(i)=\frac{1}{2}\left(\pi\csc(\pi i)-\frac{1}{i}\right)=\frac{i}{2}(1-\pi\,\mathrm{csch}(\pi))$$

$$\Omega_2=\frac{1}{\pi}\left(\int_0^\infty\frac{\sin(t)}{1+e^t}dt\right)=-\frac{i}{\pi}\left(\int_0^\infty\frac{\sinh(it)}{1+e^t}dt\right)=\frac{1}{2\pi}(1-\pi\,\mathrm{csch}(\pi))$$

$$\int_0^\infty\frac{x^2+\sin(\pi x)}{1+e^{\pi x}}dx=\int_0^\infty\frac{x^2}{1+e^{\pi x}}dx+\int_0^\infty\frac{\sin(\pi x)}{1+e^{\pi x}}dx=$$

$$\frac{3\zeta(3)}{2\pi^3}+\frac{1}{2\pi}-\frac{1}{2}\mathrm{csch}(\pi)$$

Note : $\zeta(3)\ \to$ Apery's constant

Solution 2 by Ankush Kumar Parcha-India

$$We\ have,\ \underbrace{\int_0^\infty\frac{x^2}{1+\exp(\pi x)}dx}_{\Omega_1}+\underbrace{\int_0^\infty\frac{\sin(\pi x)}{1+\exp(\pi x)}dx}_{\Omega_2}$$

$$\Omega_1=\int_0^\infty\frac{x^2}{1+\exp(\pi x)}dx\overset{\pi x\to x}{=}\frac{1}{\pi^3}\int_0^\infty\frac{x^2}{1+\exp(x)}dx\overset{Note\ section}{=}\frac{1}{\pi^3}\sum_{n\in N}(-1)^n\mathcal{L}_x\{x^2\}(n)$$

$$\underset{(2)}{\overset{Note\ Section}{\overbrace{=}}} \frac{2}{\pi^3}\sum_{n\in N}\frac{(-1)^{n-1}}{n^3} => \frac{2}{\pi^3}\eta(3) \overset{note\ \eta(s)=(1-2^{1-s}).\zeta(s)}{\triangleq} \int_0^\infty \frac{x^2}{1+\exp(\pi x)}dx = \frac{3\zeta(3)}{2\pi^3}$$

$$\Omega_2 = \int_0^\infty \frac{\sin(\pi x)}{1+\exp(\pi x)}dx \overset{\pi x\to x}{\triangleq} \frac{1}{\pi}\int_0^\infty \frac{\sin(x)}{1+\exp(x)}dx \overset{note\ section\ (1)}{\overbrace{=>}}$$

$$\frac{1}{\pi}\sum_{n\in N}(-1)^{n-1}\mathcal{L}_x\{\sin(x)\}(n) \underset{(3)}{\overset{Note\ Section}{\overbrace{=}}} \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+1} = \frac{1}{\pi}\Im\sum_{n\in N}\frac{(-1)^{n-1}}{n-i} \underset{(4)}{\overset{Note\ Section}{\overbrace{=}}}$$

$$\frac{1}{2\pi}\Im\left[\psi^{(0)}\left(1-\frac{i}{2}\right)-\psi^{(0)}\left(\frac{1}{2}-\frac{i}{2}\right)\right]$$

$$Note:\left(\ \Im\{\psi^{(0)}(z)\} = \frac{\psi^{(0)}(z)-\overline{\psi^{(0)}(z)}}{2i}\ ;\ \overline{\psi^{(0)}(z)} = \psi^{(0)}\overline{(z)}\ \right)$$

$$=> \frac{-i}{4\pi}\left[\psi^{(0)}\left(1-\frac{i}{2}\right)-\psi^{(0)}\left(1+\frac{i}{2}\right)+\psi^{(0)}\left(\frac{1}{2}+\frac{i}{2}\right)-\psi^{(0)}\left(\frac{1}{2}-\frac{i}{2}\right)\right] \overset{Note\ section\ (5,6,7)}{\overbrace{=>}}$$

$$\frac{-i}{4\pi}\left[2i+\pi\cot\left(\frac{i\pi}{2}\right)+\pi\tan\left(\frac{i\pi}{2}\right)\right] => \frac{1}{2\pi}+\frac{1}{4}\left[\tanh\left(\frac{\pi}{2}\right)-\coth\left(\frac{\pi}{2}\right)\right] \to$$

$$\Omega_2 = \int_0^\infty \frac{\sin(\pi x)}{1+\exp(\pi x)}dx = \frac{1}{2\pi}-\frac{\operatorname{csch}(\pi)}{2}$$

Put the values of Ω_1 *and* Ω_2 *in equation* $-(1)$. *We get,*

$$\Omega_1+\Omega_2 = \int_0^\infty \frac{x^2+\sin(\pi x)}{1+\exp(\pi x)}dx = \frac{1}{2\pi}-\frac{\operatorname{csch}(\pi)}{2}+\frac{3\zeta(3)}{2\pi^3}$$

Note Section:

1. *Maz Summation Identity:* $\sum_{n\in N}(-1)^{n-1}\mathcal{L}_t\{f(t)\}(n) = \int_0^\infty \frac{f(t)}{1+e^t}$

2. $\mathcal{L}_t\{t^a\}(s) = \frac{\Gamma(a+1)}{s^{a+1}}, \Re(a) > -1$

3. $\mathcal{L}_t(\sin(\omega t))(s) = \frac{\omega}{s^2+\omega^2} \quad s > |\Im(\omega)|$

4. $\sum_{n\in N}\frac{(-1)^{n-1}}{n-a} = \frac{1}{2}\left[\psi^{(0)}\left(1-\frac{a}{2}\right)-\psi^{(0)}\left(\frac{1}{2}-\frac{a}{2}\right)\right]$

5. $\psi^{(0)}\left(\frac{1}{2}+z\right)-\psi^{(0)}\left(\frac{1}{2}-z\right) = \pi\tan(\pi z)$

6. $\psi^{(0)}(1+z) = \psi^{(0)}(z)+\frac{1}{z}$

7. $\psi^{(0)}(1-z) = \psi^{(0)}(z)+\pi\cot(\pi z)$

Solution 3 by Cosghun Memmedov-Azerbaijan

$$\Omega = \int_0^\infty \frac{x^2+\sin(\pi x)}{1+e^{\pi x}}dx = \int_0^\infty \frac{x^2}{1+e^{\pi x}}dx + \int_0^\infty \frac{\sin(\pi x)}{1+e^{\pi x}}dx = M+K$$

$$M = \int_0^\infty \frac{x^2}{1+e^{\pi x}}dx \overset{\{\pi x\to x\}}{\triangleq} \frac{1}{\pi^3}\int_0^\infty \frac{x^2}{1+e^x}dx = \frac{1}{\pi^3}\int_0^\infty \frac{x^2e^{-x}}{1+e^{-x}}dx =$$

$$\frac{1}{\pi^3}\sum_{n=0}^{\infty}(-1)^n\int_0^\infty e^{-x(n+1)}x^2dx \underset{\left\{\frac{dx}{dt}=\frac{1}{n+1}\right\}}{\overset{\{x(n+1)=t\}}{\overbrace{=}}} \frac{1}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}\int_0^\infty e^{-t}t^2dt =$$

$$\frac{1}{\pi^3}\Gamma(3)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}=\frac{2}{\pi^3}\eta(3)=\frac{3}{2\pi^3}\zeta(3)$$

$$K=\int_0^\infty\frac{\sin(\pi x)}{1+e^{\pi x}}dx\overset{\{\pi x\to x\}}{\triangleq}\frac{1}{\pi}\int_0^\infty\frac{\sin(x)}{1+e^x}dx=\frac{1}{\pi}\mathcal{I}\int_0^\infty\frac{e^{ix}}{1+e^x}dx=$$

$$\frac{1}{\pi}\mathcal{I}\int_0^\infty\frac{e^{x(i-1)}}{e^{-x}+1}dx=\frac{1}{\pi}\mathcal{I}\left\{\sum_{n=0}^{\infty}(-1)^n\int_0^\infty e^{-x(n+1-i)}dx\right\}=\frac{1}{\pi}\mathcal{I}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1-i}\right\}=$$

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2+1}=-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2+1}=$$

$$\frac{1}{2\pi}(\pi\coth(\pi)-1)-\frac{2}{\pi}\left(\frac{\pi}{4}\coth\left(\frac{\pi}{2}\right)-\frac{1}{2}\right)=\frac{1}{2}\left(\coth(\pi)-\coth\left(\frac{\pi}{2}\right)\right)+\frac{1}{2\pi}=$$

$$\frac{1}{2}\left(\frac{e^\pi+e^{-\pi}}{e^\pi-e^{-\pi}}-\frac{e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}}{e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}}}\right)+\frac{1}{2\pi}=\frac{1}{2}\left(\frac{e^\pi+e^{-\pi}}{e^\pi-e^{-\pi}}-\frac{e^\pi+e^{-\pi}+2}{e^\pi-e^{-\pi}}\right)+\frac{1}{2\pi}=$$

$$-\frac{1}{e^\pi-e^{-\pi}}+\frac{1}{2\pi}=-\frac{\operatorname{csch}(\pi)}{2}+\frac{1}{2\pi}$$

$$\Omega=M+K=\frac{3}{2\pi^3}\zeta(3)-\frac{\operatorname{csch}(\pi)}{2}+\frac{1}{2\pi}=\frac{\pi^2+3\zeta(3)-\pi^3\operatorname{csch}(\pi)}{2\pi^3}$$