ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$I=\int_0^1 \frac{\log^2(x+1)}{x(x+1)^2}dx=\frac{\zeta(3)}{4}-\frac{\log^3(2)}{3}+\frac{\log^2(2)}{2}+\log(2)-1$$

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Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1 \frac{\log^2(x+1)}{x(x+1)^2}dx=\int_0^1 \frac{\log^2(x+1)}{x}dx-\int_0^1 \frac{\log^2(x+1)}{x+1}dx-\int_0^1 \frac{\log^2(x+1)}{(x+1)^2}dx$$

$$=I_1-I_2-I_3$$

$$I_1=\int_0^1 \frac{\log^2(x+1)}{x}dx \overset{IBP}{=} -2\int_0^1 \frac{\log(x)\log(x+1)}{x+1}dx=\frac{\zeta(3)}{4}$$

$$I_2=\int_0^1 \frac{\log^2(x+1)}{x+1}dx=\frac{\log^3(2)}{3}$$

$$I_3=\int_0^1 \frac{\log^2(x+1)}{(x+1)^2}dx=1-\log(2)-\frac{\log^2(2)}{2}$$

$$I=I_1-I_2-I_3=\frac{\zeta(3)}{4}-\frac{\log^3(2)}{3}+\frac{\log^2(2)}{2}+\log(2)-1$$