ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_1^{\infty}\frac{\ln(\sqrt{x})ln^2(1+x)}{(1+x)^2}dx$$

Proposed by Shirvan Tahirov, Cosghun Memmedov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\frac{1}{2}\int_0^{\infty}\frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx-\frac{1}{2}\int_0^{1}\frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx=\frac{1}{2}(\Omega_1-\Omega_2)$$

$$\Omega=\int_0^{\infty}\frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx\overset{\{1+x=t\}}{\cong}\int_1^{\infty}\frac{\ln(t-1)ln^2(t)}{t^2}dt\overset{\left\{\frac{1}{t}\to t\right\}}{\cong}\int_0^1\frac{\ln\left(\frac{1-t}{t}\right)ln^2(t)}{\frac{1}{t^2}.t^2}dt=$$

$$\int_0^1\ln\left(\frac{1-t}{t}\right)ln^2(t)dt=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 t^n ln^2(t)dt-\int_0^1 ln^3(t)dt=$$

$$2\sum_{n=1}^{\infty}\frac{1}{n(n+1)^3}+6=2\zeta(3)+\frac{\pi^3}{3}$$

$$\Omega_2=\int_0^{1}\frac{\ln(x)ln^2(1+x)}{(1+x)^2}dx\overset{\{1+x=t\}}{\cong}\int_1^{2}\frac{\ln(t-1)ln^2(t)}{t^2}\overset{\left\{\frac{1}{t}\to t\right\}}{\cong}\int_{\frac{1}{2}}^1\frac{\ln\left(\frac{1-t}{t}\right)ln^2(t)}{\frac{1}{t^2}.t^2}dt=$$

$$\underbrace{\int_0^1\ln\left(\frac{1-t}{t}\right)ln^2(t)\,dt}_{\Omega_1}-\int_0^{\frac{1}{2}}\ln\left(\frac{1-t}{t}\right)ln^2(t)\,dt=2\zeta(3)+\frac{\pi^3}{3}-K$$

$$K=\int_0^{\frac{1}{2}}\ln(1-t)\,ln^2(t)dt-\underbrace{\int_0^{\frac{1}{2}}ln^3(t)\,dt}_{IBP}=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{1}{2}}t^n ln^2(t)dt-$$

$$-[6tln(t)-3tln^2(t)+tln^3(t)-6t]_0^{\frac{1}{2}}=-\frac{1}{2}ln^2(2)\sum_{n=1}^{\infty}\frac{1}{n(n+1)2^n}+2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\int_0^{\frac{1}{2}}t^n\ln(t)\,dt+$$

$$+3\ln(2)+\frac{3ln^2(2)}{2}+\frac{ln^3(2)}{2}+3$$

$$=-\ln(2)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2 2^n}}_{Partial\ Sum}-\underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)^3 2^n}}_{Partial\ Sum}+ln^2(2)+$$

$$+ln^3(2)+3\ln(2)+3=2\ln(2)+\frac{7\zeta(3)}{4}+\frac{ln^3(2)}{3}+\frac{\pi^2}{6}+ln^2(2)$$

$$\Omega_2=\Omega_1-K,\ \ \Omega=\frac{1}{2}(\Omega_1-\Omega_2)=\frac{1}{2}K=\frac{7\zeta(3)}{8}+\frac{\pi^2}{12}+\frac{ln^3(2)}{6}+\frac{ln^2(2)}{2}+\ln(2)$$

Note : $\zeta(3)\to Apery's\ constant$