ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{\ln^2(x)+x\arctan(x)}{(1+x)(1+x^2)}dx=\frac{1}{64}\left(42\zeta(3)+16G+2\pi^3+\pi^2-8\pi\ln(2)\right)$$

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$$\theta=\int_0^1 \frac{\ln^2(x)+x\arctan(x)}{(1+x)(1+x^2)}dx=\int_0^1\frac{\ln^2(x)}{(1+x)(1+x^2)}dx+\int_0^1\frac{x\arctan(x)}{(1+x)(1+x^2)}dx=\theta_1+\theta_2$$

$$\theta_1=\int_0^1\frac{\ln^2(x)}{(1+x)(1+x^2)}dx=\frac{1}{2}\left(\int_0^1\frac{\ln^2(x)}{1+x}dx+\int_0^1\frac{\ln^2(x)}{1+x^2}dx-\int_0^1\frac{x\ln^2(x)}{1+x^2}dx\right)=$$

$$=\frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\ln^2(x)dx+\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln^2(x)dx-\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln^2(x)dx\right)=$$

$$=\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^3}\right)=\eta(3)+\beta(3)-\frac{1}{8}\eta(3)=\frac{7}{8}\eta(3)+\frac{\pi^3}{32}$$

$$=\frac{21\zeta(3)}{32}+\frac{\pi^3}{32}$$

$$\theta_2=\int_0^1\frac{x\arctan(x)}{(1+x)(1+x^2)}dx$$

$$\text{Using İBP method}\left\{u=\arctan(x);\frac{du}{dx}=\frac{1}{1+x^2};\ v=\int\frac{x+1-1}{(1+x)(1+x^2)}dx\right.$$

$$\left.=\frac{1}{2}\tan^{-1}(x)+\frac{1}{4}\ln(1+x^2)-\frac{1}{2}\ln(1+x)\right\}$$

$$\theta_2=\left(\arctan(x)\left(\frac{1}{2}\arctan(x)+\frac{1}{4}\ln(1+x^2)-\frac{1}{2}\ln(1+x)\right)\right)\Big|_0^1$$

$$-\frac{1}{2}\int_0^1\frac{\tan^{-1}(x)}{1+x^2}dx-\frac{1}{4}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{\ln(1+x)}{1+x^2}dx=$$

$$=\frac{\pi^2}{32}+\frac{\pi\ln(2)}{16}-\frac{\pi\ln(2)}{8}-\frac{1}{2}\int_0^{\frac{\pi}{4}}x\,dx-\frac{1}{4}\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{\ln(1+x)}{1+x^2}dx$$

$$=\frac{\pi^2}{64}-\frac{\pi\ln(2)}{16}-\frac{1}{4}J+\frac{1}{2}I$$

$$J=\int_0^1\frac{\ln(1+x^2)}{1+x^2}dx;\ \text{subtitation}\left\{\arctan(x)=t,\frac{dt}{dx}=\frac{1}{1+x^2},t\left[\frac{\pi}{4};0\right]\right\}$$

$$J=-2\int_0^{\frac{\pi}{4}}\ln(\cos(t))dt;\ \ \text{note:}\left\{\text{Fourier series }\ln(\cos(x))\right.$$

$$\left.=-\ln(2)-\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{n}\right\}$$

$$J=2\ln(2)\int_0^{\frac{\pi}{4}}dt+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nt)dt=\frac{\pi}{2}\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2}=\frac{\pi}{2}\ln(2)-G$$

$$I=\int_0^1 \frac{\ln(1+x)}{1+x^2}dx;\ subtitation:\left\{\frac{1-t}{1+t}=x, \frac{dx}{dt}=-\frac{2}{(1+t)^2}, t[0;1]\right\}$$

$$I=\int_0^1 \frac{\ln\left(\frac{2}{1+t}\right)}{1+t^2}dt=\ln(2)\int_0^1 \frac{1}{1+t^2}dt-\int_0^1 \frac{\ln(1+t)}{1+t^2}dt=\frac{\pi}{4}\ln(2)-I;\ \ 2I=\frac{\pi}{4}\ln(2)\ \ I=\frac{\pi}{8}\ln(2)$$

$$\theta_2=\frac{\pi^2}{64}-\frac{\pi\ln(2)}{16}-\frac{1}{4}J+\frac{1}{2}I=\frac{\pi^2}{64}-\frac{\pi}{16}\ln(2)-\frac{\pi}{8}\ln(2)+\frac{G}{4}+\frac{\pi}{16}\ln(2)=\frac{\pi^2}{64}-\frac{\pi}{8}\ln(2)+\frac{G}{4}$$

$$\int_0^1 \frac{ln^2(x)+x\arctan(x)}{(1+x)(1+x^2)}dx=\theta_1+\theta_2=\frac{21}{32}\zeta(3)+\frac{\pi^3}{32}+\frac{\pi^2}{64}-\frac{\pi}{8}\ln(2)+\frac{G}{4}$$

$$=\frac{1}{64}\left(42\zeta(3)+2\pi^3+\pi^2+16G-8\pi\ln(2)\right)$$