ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \int_1^\infty \frac{y ln(1-y^2) ln^2(1+x)}{x^2(1+x)} dxdy$$

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$$\int_0^1 \int_1^\infty \frac{y ln(1-y^2) ln^2(1+x)}{x^2(1+x)} dxdy = \int_0^1 y ln(1-y^2) dy \underbrace{\int_1^\infty \frac{ln^2(1+x)}{x^2(1+x)} dx}_{x\to\frac{1}{x}} =$$

$$-\frac{1}{2}\int_0^1 ln(1-y^2) d(1-y^2) \int_0^1 \frac{x ln^2\left(1+\frac{1}{x}\right)}{1+x} dx = -\frac{1}{2}\int_0^1 \frac{x ln^2\left(1+\frac{1}{x}\right)}{1+x} dx = -\frac{1}{2} I$$

$$* \; Consider: \int_0^1 \frac{x ln^2(1+x)}{1+x} dx \overset{IBP}{\triangleq} 2 - \frac{1}{3} ln^3(2) + 2ln^2(2) - 4\ln(2)$$

$$But: \int_0^1 \frac{x ln^2(1+x)}{1+x} dx = \int_0^1 \frac{x(\ln(x) + \ln(1+\frac{1}{x}))^2}{1+x} dx = I + \int_0^1 \frac{x ln^2(x)}{1+x} dx +$$

$$2\int_0^1 \frac{x ln(x) \ln(1+\frac{1}{x})}{1+x} dx = I + 2\int_0^1 \frac{x ln(x) \ln(1+x)}{1+x} dx - \int_0^1 \frac{x ln^2(x)}{1+x} dx$$

$$1) \int_0^1 \frac{x ln^2(x)}{1+x} dx = \sum_{n=0}^\infty (-1)^n \int_0^1 x^{n+1} ln^2(x) dx = 2n$$

$$= 2\sum_{n=0}^\infty (-1)^n \frac{1}{(n+2)^3} = 2 - \frac{3}{2}\zeta(3)$$

$$2) \int_0^1 \frac{x ln(x) \ln(1+x)}{1+x} dx = -\int_0^1 x ln(x) \sum_{n=1}^\infty (-1)^n H_n x^n dx =$$

$$-\sum_{n=1}^\infty (-1)^n H_n \int_0^1 x^{n+1} \ln(x)\, dx = \sum_{n=1}^\infty (-1)^n H_n \frac{1}{(n+2)^2} =$$

$$\sum_{n=1}^\infty (-1)^n \left(H_{n+2} - \frac{1}{n+2} - \frac{1}{n+1}\right) \frac{1}{(n+2)^2} = \sum_{n=1}^\infty (-1)^n \frac{H_{n+2}}{(n+2)^2} - \underbrace{\sum_{n=1}^\infty (-1)^n \frac{1}{(n+2)^3}}_{=\frac{7}{8}-\frac{3}{4}\zeta(3)}$$

$$-\underbrace{\sum_{n=1}^{\infty}(-1)^n\frac{1}{(n+1)(n+2)^2}}_{=\frac{\pi^2}{12}+2\ln(2)-\frac{9}{4}}=\frac{3}{4}\zeta(3)+\frac{11}{8}-\frac{\pi^2}{12}-2\ln(2)+\sum_{n=3}^{\infty}(-1)^n\frac{H_n}{n^2}=\frac{3}{4}\zeta(3)+\frac{11}{8}-$$

$$-\frac{\pi^2}{12}-2\ln(2)+\left(\underbrace{\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^2}}_{=-\frac{5}{8}\zeta(3)}+1-\frac{3}{8}\right)=\frac{\zeta(3)}{8}-\frac{\pi^2}{12}+2-2\ln(2)$$

$$=>2-\frac{1}{3}ln^3(2)+2ln^2(2)-4\ln(2)$$

$$=I+2\left(\frac{\zeta(3)}{8}-\frac{\pi^2}{12}+2-2\ln(2)\right)-\left(2-\frac{3}{2}\zeta(3)\right)$$

$$=>I=-\frac{7}{4}\zeta(3)+\frac{\pi^2}{6}-\frac{1}{3}ln^3(2)+2ln^2(2)$$

$$\int_0^1\int_1^{\infty}\frac{yln(1-y^2)ln^2(1+x)}{x^2(1+x)}dxdy=-\frac{1}{2}I=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}+\frac{1}{6}ln^3(2)-ln^2(2)=$$

$$\frac{1}{24}\left(21\zeta(3)+4ln^3(2)-2\pi^2-24ln^2(2)\right)$$

Note : $\zeta(3)\rightarrow$ *Apery's constant*