ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_1^{\infty}\frac{\log^2(x)}{(x+1)(x+2)}dx=\frac{1}{12}\left(-6\mathrm{Li}_3\left(\frac{1}{4}\right)+3\varsigma(3)+8\log^3(2)+\pi^2\log(4)\right)$$

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$$I=\int_1^{\infty}\frac{\log^2(x)}{(x+1)(x+2)}dx=\int_0^{1}\frac{\log^2(x)}{(x+1)(2x+1)}dx=2\int_0^1\frac{\log^2(x)}{1+2x}dx-\int_0^1\frac{\log^2(x)}{1+x}dx$$

$$=2\sum_{k=0}^{\infty}(-1)^k2^k\int_0^1 x^k\log^2(x)\,dx-\sum_{k=0}^{\infty}(-1)^k\int_0^1 x^k\log^2(x)\,dx$$

$$=-2\sum_{k=0}^{\infty}\frac{(-2)^{k+1}}{(k+1)^3}-2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3}=2\mathrm{Li}_3(-2)-2\eta(3)$$

$$=-2\left\{-\frac{1}{3}\log^3(2)-\frac{\pi^2}{12}\log(2)+\frac{1}{4}\mathrm{Li}_3\left(\frac{1}{4}\right)-\frac{7}{8}\varsigma(3)\right\}-\frac{3}{2}\varsigma(3)$$

$$=-\frac{1}{2}\mathrm{Li}_3\left(\frac{1}{4}\right)+\frac{\pi^2}{6}\log(2)+\frac{2}{3}\log^3(2)+\frac{1}{4}\varsigma(3)$$

$$=\frac{1}{12}\left(-6\mathrm{Li}_3\left(\frac{1}{4}\right)+3\varsigma(3)+8\log^3(2)+\pi^2\log(4)\right)$$

NOTE:
$$\begin{cases}\mathrm{Li}_3(z)+\mathrm{Li}_3(-z)=\frac{1}{4}\mathrm{Li}_3(z^2)\\ \mathrm{Li}_3(-z)-\mathrm{Li}_3\left(-\frac{1}{z}\right)=-\frac{1}{6}\log^3(z)-\frac{\pi^2}{6}\log(z)\end{cases}$$

$$\mathrm{Li}_3(-2)=-\frac{1}{3}\log^3(2)-\frac{\pi^2}{12}\log(2)+\frac{1}{4}\mathrm{Li}_3\left(\frac{1}{4}\right)-\frac{7}{8}\varsigma(3)$$