ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \int_0^1 \frac{x(ln\sqrt{1+x^2} + arctan^2(x))}{1+x^2}dx$$

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Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^1 \frac{x(ln\sqrt{1+x^2} + arctan^2(x))}{1+x^2}dx = \Omega_1 + \Omega_2 \begin{cases} \Omega_1 = \int_0^1 \frac{xln\sqrt{1+x^2}}{1+x^2}dx \\ \Omega_2 = \int_0^1 \frac{xarctan^2(x)}{1+x^2}dx \end{cases}$$

$$\Omega_1 = \int_0^1 \frac{xln\sqrt{1+x^2}}{1+x^2}dx \overset{IBP}{=} \frac{1}{2}\left[\frac{1}{2}ln^2(1+x^2)\right]_0^1 - \frac{1}{2}\int_0^1 \frac{xln(1+x^2)}{1+x^2}dx = \frac{1}{4}ln^2(2) - \Omega_1$$

$$2\Omega_1 = \frac{1}{4}ln^2(2) \lll=\ggg \Omega_1 = \frac{1}{8}ln^2(2)$$

$$\Omega_2 = \int_0^1 \frac{xarctan^2(x)}{1+x^2}dx \overset{IBP}{=} \left[\frac{1}{2}arctan^2(x)\ln(1+x^2)\right]_0^1$$

$$-\int_0^1 \frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx =$$

$$\frac{\pi^2}{32}\ln(2) - \int_0^1 \frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx \lll=\ggg \left\{\arctan(x) = t;\ \frac{dt}{dx} = \frac{1}{1+x^2}; x = \tan(t)\, t\left[\frac{\pi}{4};0\right]\right\}$$

$$\Omega_2 = \frac{\pi^2}{32}\ln(2) - \int_0^{\frac{\pi}{4}} tln\left(1+tan^2(t)\right)dt = \frac{\pi^2}{32}\ln(2) + 2\int_0^{\frac{\pi}{4}} tln(\cos(t))dt$$

$$= \frac{\pi^2}{32}\ln(2) + 2I$$

Note :

$$\left\{We\ know \to \ln(\cos(t)) = -\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nt)}{n}\quad Fourier\ series\right\}$$

$$I = \int_0^{\frac{\pi}{4}} tln(\cos(t))dt = -\ln(2)\int_0^{\frac{\pi}{4}} tdt - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} tcos(2nt)dt = -\frac{\pi^2}{32}\ln(2) -$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\cos(2nt)}{4n^2} + \frac{tsin(2nt)}{2n}\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{\pi^2}{32}\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi\sin\left(\frac{\pi n}{2}\right)}{8n} + \frac{\cos\left(\frac{\pi n}{2}\right)}{4n^2} - \frac{1}{4n^2}\right] =$$

$$-\frac{\pi^2}{32}\ln(2)-\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right)-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\left(\frac{\pi n}{2}\right)+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=$$

$$-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}+\frac{1}{4}\left(\frac{1}{2^3}\left(1-\frac{1}{2^3}+\frac{1}{3^3}-\cdots\right)\right)-\frac{1}{4}(1-2^{1-3})\zeta(3)=$$

$$-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{8}G+\frac{3\zeta(3)}{16}+\frac{1}{32}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}=\frac{\pi}{8}G-\frac{\pi^2}{32}\ln(2)-\frac{3\zeta(3)}{16}+\frac{3\zeta(3)}{128}=$$

$$\frac{\pi}{8}G-\frac{\pi^2}{32}\ln(2)-\frac{21\zeta(3)}{128}$$

$$\Omega_2=\frac{\pi^2}{32}\ln(2)+2I=\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{\pi^2}{16}\ln(2)-\frac{21\zeta(3)}{64}=\frac{\pi}{4}G-\frac{\pi^2}{32}\ln(2)-\frac{21\zeta(3)}{64}$$

$$\int_0^1\frac{x(ln\sqrt{1+x^2}+\ arctan^2(x))}{1+x^2}dx=\ \Omega_1+\Omega_2=\frac{\pi}{4}G-\frac{\pi^2}{32}\ln(2)-\frac{21\zeta(3)}{64}+\frac{1}{8}ln^2(2)$$

Solution 2 by Ankush Kumar Parcha-India

$$We\ \ have,\ \underbrace{\frac{1}{2}\int_0^1\frac{xln(1+x^2)}{1+x^2}dx}_{X}+\underbrace{\int_0^1\frac{xarctan^2(x)}{1+x^2}dx}_{Y}\qquad (1)$$

$$X=\underbrace{\frac{1}{2}\int_0^1\frac{xln(1+x^2)}{1+x^2}dx}\overset{x^2\to x}{\triangleq}\frac{1}{4}\int_0^1\frac{\ln(1+x)}{1+x}dx=>\left(\frac{ln^2(1+x)}{8}\right)_0^1$$

$$X=\underbrace{\frac{1}{2}\int_0^1\frac{xln(1+x^2)}{1+x^2}dx}=\frac{ln^2(2)}{8}$$

$$Y=\underbrace{\int_0^1\frac{xarctan^2(x)}{1+x^2}dx}\overset{tan^{-1}(x)\to x}{\triangleq}\int_0^{\frac{\pi}{4}}x^2\tan(x)\,dx\overset{IBP}{\triangleq}-\left(x^2\int\frac{d}{dx}lncos(x)dx\right)_0^{\frac{\pi}{4}}+$$

$$2\int_0^{\frac{\pi}{4}}xlncos(x)d\,x\ \underset{(1)}{\overset{Note\ section}{\underbrace{\triangleq}}}\ \frac{\pi^2}{32}ln(2)-2\ln(2)\int_0^{\frac{\pi}{4}}xdx$$

$$-2\sum_{n\in N}\frac{(-1\text{-})^n}{n}\int_0^{\frac{\pi}{4}}xcos(2nx)dx=$$

$$\overset{IBP}{\triangleq}\frac{\pi^2}{32}ln(2)-\frac{\pi^2\ln(2)}{16}$$

$$-2\sum_{n\in N}\frac{(-1\text{-})^n}{n}\left[\left(\frac{xsin(2nx)}{2n}\right)_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{\sin(2nx)}{2n}dx\right]=-\frac{\pi^2}{32}ln(2)-$$

$$2\sum_{n\in N}\frac{(-1\text{-})^n}{n}\left[\frac{\pi}{8n}\sin\left(\frac{\pi n}{2}\right)+\left(\frac{\cos(2nx)}{4n^2}\right)_0^{\frac{\pi}{4}}\right]=-\frac{\pi^2}{32}ln(2)-\frac{\pi}{4}\underbrace{\sum_{n\in N}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2}}_{n\to 2n+1}-$$

$$\underbrace{\frac{1}{2}\sum_{n\in N}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3}}_{n\to 2n}+\frac{1}{2}\sum_{n\in N}\frac{(-1)^{n+1}}{n^3}\overset{Note\ section}{\underbrace{=}_{(2)}}-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}\sum_{n\in N\cup\{0\}}\frac{(-1)^n}{(2n+1)^2}+$$

$$\frac{1}{16}\sum_{n\in N}\frac{(-1)^{n+1}}{n^3}-\frac{1}{2}\sum_{n\in N}\frac{(-1)^{n+1}}{n^3}=-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}G-\frac{7}{16}\eta(3)$$

$$X+Y=\int_0^1\frac{x(ln\sqrt{1+x^2}+\ arctan^2(x))}{1+x^2}dx=\frac{\pi}{4}G-\frac{21}{64}\zeta(3)+\frac{ln^2(2)}{8}-\frac{\pi^2\ln(2)}{32}$$

Note section **:**

1. $\ln(2\cos(x))=\sum_{n\in N}\frac{(-1)^{n+1}}{n}\cos(2nx),\ \ |x|<\frac{\pi}{2}$
2. $\sum_{n\in N\cup\{0\}}\frac{(-1)^n}{(2n+1)^2}=G\ \ (Catalan's\ constant)$
3. $\eta(s)=(1-2^{1-s})\ .\ \zeta(3)$