ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \int_0^1 \frac{\ln(1+x^2)\,(\arctan(x)+x)}{1+x^2}\,dx$$

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$$\Omega = \int_0^1 \frac{\ln(1+x^2)\arctan(x)}{1+x^2}\,dx + \int_0^1 \frac{x\ln(1+x^2)}{1+x^2}\,dx = \Omega_1 + \Omega_2$$

$$\Omega_1 = \int_0^1 \frac{\ln(1+x^2)\tan^{-1}(x)}{1+x^2}\,dx, \left\{\tan^{-1}(x) = t\,, dt = \frac{dx}{1+x^2}\,, t\left[\frac{\pi}{4};0\right]\right\}$$

$$\Omega_1 = \int_0^{\frac{\pi}{4}} t\ln(1+\tan^2\ (t))\,dt = -2\int_0^{\frac{\pi}{4}} t\ln\big(\cos(t)\big)\,dt$$

$$\left\{\textit{Fourier series of } \ln\big(\cos(z)\big) = -\ln(2) - \sum_{n=1}^{\infty} \frac{(-1)^n\cos(2nz)}{n}\right\}$$

$$\Omega_1 = 2\ln^2(2)\int_0^{\frac{\pi}{4}} t\,dt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} t\cos(2nt)\,dt = \frac{\pi^2\ln(2)}{16} +$$

$$+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{t\sin(2nt)}{2n}\Big]_0^{\frac{\pi}{4}} - \frac{1}{2n}\int_0^{\frac{\pi}{4}}\sin(2nt)\,dt\right] = \pi^2\frac{\ln(2)}{16} + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2} +$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left[\frac{\cos(2nt)}{2n}\right]_0^{\frac{\pi}{4}} =$$

$$= \pi^2\frac{\ln(2)}{16} + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} =$$

$$= \frac{\pi^2\ln(2)}{16} - \frac{\pi}{4}G - \frac{3}{64}\zeta(3) + \frac{3}{8}\zeta(3) = \frac{\pi^2\ln(2)}{16} - \frac{\pi}{4}G + \frac{21}{64}\zeta(3)$$

$$\Omega_2 = \int_0^1 \frac{x\ln(1+x^2)}{1+x^2}\,dx, \qquad \{x^2 = t, dt = 2x\,dx, \qquad t[1;0]\}$$

$$\Omega_2 = \int_0^1 \frac{\ln(1+t)}{1+t}\,dt = \frac{1}{2}\left[\frac{\ln^2(1+t)}{2}\right]_0^1 = \frac{\ln^2(2)}{4}$$

$$\int_0^1 \frac{\ln(1+x^2)(\tan^{-1}(x)+x)}{1+x^2}\,dx = \Omega_1 + \Omega_2 = \frac{\pi^2\ln(2)}{16} - \frac{\pi}{4}G + \frac{21}{64}\zeta(3) + \frac{\ln^2(2)}{4}$$