ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_1^\infty \frac{\sqrt{x}\ln(x)+y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy$$

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Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_1^\infty \frac{\sqrt{x}\ln(x)}{(1+x)^2}dxdy+\int_0^1\int_1^\infty \frac{y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\int_1^\infty \frac{\sqrt{x}\ln(x)}{(1+x)^2}dxdy, substitution\left\{\frac{1}{x}=t,dt=-t^2dx,t[0;1]\right\}$$

$$\Omega_1=-\int_0^1\frac{\ln(x)}{\sqrt{x}(1+x)^2}dx=\sum_{n=1}^\infty n(-1)^n\int_0^1 x^{n-\frac{3}{2}}\ln(x)\,dx=-4\sum_{n=1}^\infty\frac{(-1)^n n}{(2n-1)^2}=$$
$$=-2\left(\sum_{n=1}^\infty\frac{(-1)^n}{(2n-1)}+\sum_{n=1}^\infty\frac{(-1)^n}{(2n-1)^2}\right)=-2\left(-G-\frac{\pi}{4}\right)=2G+\frac{\pi}{2}$$

$$\Omega_2=\int_0^1\int_1^\infty \frac{y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy=-[\frac{1}{1+x}]_1^\infty\int_0^1 y\ln\left(\arccos^2(1-y)\right)dy=$$
$$=\int_0^1\ln(\arccos(y))\,dy-\int_0^1 y\ln\left(\arccos^2(y)\right)dy=I-J$$

$$I=\int_0^1\ln(\arccos(y))\,dy,\left\{\cos^{-1}(y)=t,dt=-\frac{dy}{\sqrt{1-y^2}}=-\frac{dy}{\sin(t)},t\left[0;\frac{\pi}{2}\right]\right\}$$
$$I=\int_0^{\frac{\pi}{2}}\sin(t)\ln(t)\,dt=[-\ln(t)\cos(t)]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\frac{\cos(t)-1+1}{t}dt=Ci\left(\frac{\pi}{2}\right)-\gamma$$

$$J=\int_0^1 y\ln(\arccos(y))\,dy=\int_0^{\frac{\pi}{2}}\sin(t)\cos(t)\ln(t)\,dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2t)\ln(t)\,dt=$$
$$=\frac{1}{2}[-\frac{\ln(t)\cos(2t)}{2}]_0^{\frac{\pi}{2}}+\frac{1}{4}\int_0^{\frac{\pi}{2}}\underset{\{2t\to t\}}{\frac{\cos(2t)-1}{t}}dt+\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1}{t}dt=$$
$$=\frac{1}{4}\left(\ln\left(\frac{\pi}{2}\right)+\ln(0)\right)+\frac{1}{4}(Ci(\pi)-\gamma-\ln(\pi))+\frac{1}{4}\ln\left(\frac{\pi}{2}\right)-\frac{1}{4}\ln(0)=$$

$$=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)+\frac{1}{4}Ci(\pi)-\frac{\gamma}{4}-\frac{\ln(\pi)}{4}=\frac{1}{4}Ci\left(\frac{\pi}{2}\right)-\frac{1}{4}\ln\left(\frac{4}{\pi}\right)-\frac{\gamma}{4}$$

$$\Omega_2=I-J=Ci\left(\frac{\pi}{2}\right)-\gamma-\frac{1}{4}Ci(\pi)+\frac{1}{4}\ln\left(\frac{4}{\pi}\right)+\frac{\gamma}{4}=Ci\left(\frac{\pi}{2}\right)-\frac{1}{4}Ci(\pi)-\frac{3\gamma}{4}+\frac{1}{4}\ln\left(\frac{4}{\pi}\right)$$

$$\int_0^1\int_1^\infty\frac{\sqrt{x}\ln(x)+y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy=\Omega_1+\Omega_2=$$

$$=2G+\frac{\pi}{2}+Ci\left(\frac{\pi}{2}\right)-\frac{1}{4}Ci(\pi)-\frac{3\gamma}{4}+\frac{1}{4}\ln\left(\frac{4}{\pi}\right)$$

$$note\left\{cosine\ integral: Ci(x)=\gamma+\ln(x)+\int_0^x\frac{\cos(z)-1}{z}dz\right\}$$

Solution 2 by Exodo Halcalias-Angola

$$H=\int_1^\infty\frac{\sqrt{x}\ln(x)}{(1+x)^2}dx+\int_0^1 yln(\arccos(1-y))^2dy$$

$$H_1=\int_1^\infty\frac{\sqrt{x}\ln(x)}{(1+x)^2}dx\overset{x\to 1/x}{\triangleq}\int_0^1\frac{\ln(\frac{1}{x})}{\sqrt{x}(1+x)^2}dx\overset{IBP}{\triangleq}\int_0^1\frac{\arctan(\sqrt{x})}{x}dx+\int_0^1\frac{\sqrt{x}}{(1+x)x}dx=$$

$$2\int_0^1\frac{\arctan(x)}{x}dx+1\int_0^1\frac{1}{1+x^2}dx=2G+\frac{\pi}{2}$$

$$H_2=\int_0^1 yln(\arccos(1-y))^2dy=2\int_0^1 yln(\arccos(1-y))dy\overset{1-y\to y}{\triangleq}$$

$$2\left(\int_0^1\ln(\arccos(y))dy-\int_0^1 yln(\arccos(y))dy\right)$$

$$E_1=\int_0^1\ln(\arccos(y))dy\overset{\arccos(y)\to y}{\triangleq}\int_0^{\frac{\pi}{2}}\sin(y)\ln(y)\,dy\overset{IBP}{\triangleq}\left[\ln(x)\,(1-\cos(y))\right]_0^{\frac{\pi}{2}}+$$

$$\int_0^{\frac{\pi}{2}}\frac{\cos(y)-1}{y}dy=\ln\left(\frac{\pi}{2}\right)+\int_0^{\frac{\pi}{2}}\frac{\cos(y)-1}{y}dy$$

$$Ci(z)-\gamma=\ln(z)+\int_0^z\frac{\cos(y)-1}{y}dy$$

$$E_1=Ci\left(\frac{\pi}{2}\right)-\gamma$$

$$E_2=\int_0^1 yln(\arccos(y))dy\overset{\arccos(y)\to y}{\triangleq}\int_0^{\frac{\pi}{2}}\ln(y)\,sin\,(y)\cos(y)\,dy=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(y)\sin(2y)\,dy$$

$$\overset{2y\to y}{\triangleq}\frac{1}{4}\int_0^\pi\ln(y/2)\sin(y)\,dy=\frac{1}{4}\int_0^\pi\ln(y)\sin(y)\,dy-\frac{1}{4}\ln(2)\int_0^\pi\sin(y)\,dy\overset{IBP}{\triangleq}$$

$$\frac{1}{4}\left\{\left[\ln(y)(1-\cos(y))\right]_0^\pi+\int_0^\pi\frac{\cos(y)-1}{y}dy\right\}-\frac{2}{4}\ln(2)=$$

Recalling that: $Ci(z)-\gamma=\ln(z)+\int_0^z\frac{\cos(y)-1}{y}dy$

$$E_2=\frac{1}{4}(Ci(\pi)-\gamma+\ln(\pi)-\ln(4))=\frac{1}{4}(Ci(\pi)-\gamma+\ln(\frac{\pi}{4})$$

$$H_2=2(E_1-E_2)=2Ci\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}-\frac{3}{2}\gamma-\frac{1}{2}\ln(\frac{\pi}{4})$$

$$H=H_1+H_2=2G+2Ci\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}-\frac{3}{2}\gamma+\frac{1}{2}\ln\left(\frac{4}{\pi}\right)+\frac{\pi}{2}$$

$$\int_0^1\int_1^\infty\frac{\sqrt{x}\ln(x)+yln(\arccos(1-y))^2}{(1+x)^2}dxdy$$

$$=2G+2Ci\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}+\frac{1}{2}(\pi-3_\gamma+\ln\left(\frac{4}{\pi}\right))$$