ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_1^\infty \frac{yx^2(\ln(x)+1)^2 ln^2(1+y^2)}{(x^2+1)^2(y^2+1)}dxdy$$

Proposed by Shirvan Tahirov, Abbaszade Yusif-Azerbaijan

Solution 1 by Djamel Arrouche-Algeria

$$\int_1^\infty \frac{x^2(\ln(x)+1)^2}{(x^2+1)^2}dx\,.\,\int_0^1\frac{yln^2(1+y^2)}{(y^2+1)}dy=\int_0^1\frac{(1-\ln(x))^2}{(x^2+1)^2}dx.\int_0^1\frac{ln^2(1+t)}{1+t}.\frac{dt}{2}=$$

$$=\frac{1}{6}ln^3(2).\int_0^1\frac{(ln^2(x)-2\ln(x)+1)}{(x^2+1)^2}dx=\frac{1}{6}ln^3(2)\,.\,\Omega$$

$$\Omega=\int_0^1\frac{ln^2(x)}{(x^2+1)^2}dx-2\int_0^1\frac{\ln(x)}{(x^2+1)^2}dx+\int_0^1\frac{dx}{(x^2+1)^2}$$

$$\int_0^1\frac{dx}{(x^2+1)^2}=\int_0^{\frac{\pi}{4}}\frac{dy}{1+tan^2(y)}=\int_0^{\frac{\pi}{4}}cos^2(y)dy=\int_0^{\frac{\pi}{4}}\frac{1+\cos(2y)}{2}dy=\frac{\pi}{8}+\frac{1}{4}$$

$$\frac{1}{(x^2+1)^2}=\sum_{n\ge1}n(-1)^{n-1}x^{2n-2}$$

$$-2\int_0^1\frac{\ln(x)}{(x^2+1)^2}dx$$

$$=\sum_{n\ge1}n(-1)^{n-1}\int_0^1 x^{2n-2}\ln(x)\,dx=-2\sum_{n\ge1}n(-1)^{n-1}\left[-\frac{1}{(2n-1)^2}\right]=$$

$$\sum_{n\ge1}\frac{n(-1)^{n-1}(2n-1+1)}{(2n-1)^2}=\sum_{n\ge1}\frac{(-1)^{n-1}}{(2n-1)^2}+\sum_{n\ge1}\frac{(-1)^{n-1}}{(2n-1)^2}=G+\frac{\pi}{4}$$

$$\int_0^1\frac{ln^2(x)}{(x^2+1)^2}dx=\int_0^1 ln^2(x)\sum_{n\ge1}n(-1)^{n-1}x^{2n-2}\,dx=\sum_{n\ge1}n(-1)^{n-1}.\frac{1}{(2n-1)^3}=$$

$$\sum_{n\ge1}\frac{(-1)^{n-1}}{(2n-1)^3}+\sum_{n\ge1}\frac{(-1)^{n-1}}{(2n-1)^2}=G+\frac{\pi^3}{32}$$

$$\Omega=\frac{\pi}{8}+\frac{1}{4}+G+\frac{\pi}{4}+G+\frac{\pi^3}{32}=\frac{1}{32}(64G+\pi^3+12\pi+8)$$

Solution 2 by Quadi Faruk Temitope-Nigeria

$$I=\underbrace{\int_1^\infty\frac{x^2(\ln(x)+1)^2}{(x^2+1)^2}dx}_{A}.\underbrace{\int_0^1\frac{yln^2(1+y^2)}{(y^2+1)}dy}_{B}$$

ROMANIAN MATHEMATICAL MAGAZINE

$$A=\int_1^{\infty}\frac{x^2(\ln(x)+1)^2}{(x^2+1)^2}dx\underbrace{\overbrace{=}^{x\to\frac{1}{x}}}_{dx=-\frac{dx}{x^2}}=\int_0^1\frac{[-\ln(x)+1]^2}{x^2\left(\frac{x^2+1}{x^2}\right)^2}\frac{dx}{x^2}=\int_0^1\frac{[-\ln(x)+1]^2}{(x^2+1)^2}dx=$$

$$[-\ln(x)+1]^2\left[\frac{\arctan(x)}{2}+\frac{x}{2(x^2+1)}\right]_0^1+\int_0^1\frac{dx}{x^2+1}+\int_0^1\frac{\arctan(x)}{x}dx$$

$$-\int_0^1\frac{\ln(x)}{x^2+1}dx-\int_0^1\frac{\ln(x)\arctan(x)}{x}dx$$

$$A=\frac{\pi}{8}+\frac{1}{4}+\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n}dx+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1x^{2n-1+1}dx$$

$$-\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n}\ln(x)dx-$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1x^{2n+1-1}\ln(x)dx=\frac{\pi}{8}+\frac{1}{4}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=2G+\frac{\pi^3}{32}+\frac{3\pi}{8}+\frac{1}{4}$$

$$B=\int_0^1\frac{yln^2(1+y^2)}{(y^2+1)}dy\overset{IBP}{\cong}\frac{1}{2}ln^3(y^2+1)\Big|_0^1-2\underbrace{\int_0^1\frac{yln^2(1+y^2)}{(y^2+1)}dy}_{B}$$

$$B+2B=\frac{1}{2}ln^3(y^2+1)\Big|_0^1\ ,\ 3B=\frac{1}{2}ln^3(2)\quad B=\frac{1}{6}ln^3(2)\quad This\quad I=A\,.\,B$$

$$I=\frac{1}{6}ln^3(2)\left(2G+\frac{\pi^3}{32}+\frac{3\pi}{8}+\frac{1}{4}\right)=\frac{1}{6}ln^3(2)\,.\,\frac{1}{32}(64G+\pi^3+12\pi+8)=$$

$$\frac{1}{192}ln^3(2)(64G+\pi^3+12\pi+8)\quad Note:G\to Catalan's\ constant\ \ldots$$