ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 (x\ln\left(\cos^{-1}(1-x^2)\right) + (\ln\left(\frac{x}{x+1}\right) + 1)^2)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Djamel Arrouche-Algeria

$$\int_0^1 x\ln\left(\cos^{-1}(1-x^2)\right)dx = \Omega_1 \quad 1-x^2 = y\,;\ \Omega = \frac{1}{2}\int_0^1 \ln\left(\cos^{-1}(y)\right)dy\,;\ y = \cos(s)$$

$$\Omega = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(s)\sin(s)\,ds \overset{IBP}{\triangleq} u = \ln(s), v' = \sin(s)\,; u' = \frac{1}{s}\ \ v(s) = 1-\cos(s)$$

$$\Omega = \frac{1}{2}[(1-\cos(s)\ln(s)]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(s)}{s}ds = \frac{1}{2}\ln\left(\frac{\pi}{2}\right) - \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(s)}{s}ds$$

$$Ci(z) = \ln(z) + \gamma + \int_0^z \frac{\cos(x)-1}{x}dx$$

$$\Omega_1 = \frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right) - \gamma\right]$$

$$\Omega_2 = \int_0^1 (\ln\left(\frac{x}{x+1}\right) + 1)^2 dx\,;\ \frac{x}{1+x} = y \to x = \frac{y}{1-y} \Rightarrow dx = \frac{dy}{(1-y)^2}$$

$$A = \int_0^{\frac{1}{2}} (\ln(y)+1)^2 . \frac{dy}{(1-y)^2} = \int_0^{\frac{1}{2}} \frac{ln^2(y)}{(1-y)^2} + 2\int_0^{\frac{1}{2}} \frac{ln(y)}{(1-y)^2} + \frac{dy}{(1-y)^2}$$

$$\lim_{x\to 0}\left[\frac{ln^2(y)}{1-y}\right]_x^{\frac{1}{2}} - \int_x^{\frac{1}{2}} \frac{2\ln(y)}{y(1-y)}dy + 2\left[\frac{\ln(y)}{1-y}dy\right]_x^{\frac{1}{2}} - \int_x^{\frac{1}{2}} \frac{2}{y(1-y)}dy + 1 = 2ln^2(2) +$$

$$\lim_{x\to 0} -\frac{ln^2(x)}{1-x} - ln^2\left(\frac{1}{2}\right) + ln^2(x) - 4\ln(2) - \frac{2\ln(x)}{1-x} - 2\ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{1}{2}\right) + 2\ln(x) +$$

$$2\ln(1-x) - 2\int_0^{\frac{1}{2}} \frac{\ln(y)}{1-y}dy = ln^2(2) - 4\ln(2) + 1 + 2\int_0^{\frac{1}{2}} \frac{\ln(1-(1-y))}{1-y}d(1-y) +$$

$$\lim_{x\to}\left[-\frac{ln^2(x)}{1-x} + ln^2(x) + 2\ln(x) - \frac{2\ln(x)}{1-x} + 2kn(1-x)\right]x = 0^{\cdot}, ''\lim_{x\to 0} xln^n(x) = 0''$$

$$= ln^2(2) - 4\ln(2) + 1 - 2\left[Li_2\left(\frac{1}{2}\right) - Li_2(1)\right] \quad Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{ln^2(2)}{2},$$

$$Li_2(1) = \frac{\pi^2}{6}$$

$$\Omega_2 = 2ln^2(2) - 4\ln(2) + 1 + \frac{\pi^2}{6}$$

$$\Omega = \Omega_1 + \Omega_2 = \frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right) - \gamma\right] + 2ln^2(2) - 4\ln(2) + 1 + \frac{\pi^2}{6}$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1\left(x\ln\left(\cos^{-1}(1-x^2)\right)+\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2\right)dx=\underbrace{\int_0^1 x\ln\left(\cos^{-1}(1-x^2)\right)dx}_{A}+$$

$$+\underbrace{\int_0^1\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2dx}_{B}$$

$$A=\int_0^1 x\ln\left(\cos^{-1}(1-x^2)\right)dx\underset{dx=\frac{dp}{2x}}{\overset{x^2=p}{=}}\int_0^1 x\ln\left(\cos^{-1}(1-p)\right)\frac{dp}{2x}$$

$$=\frac{1}{2}\int_0^1 \ln\left(\cos^{-1}(1-p)\right)dp=$$

$$\frac{1}{2}Ci\left(\cos^{-1}(1-p)\right)+\frac{1}{2}(p-1)\ln\left(\cos^{-1}(1-x^2)\right)\Big|_0^1=$$

$$=\frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right)-\gamma\right]$$

$$B=\int_0^1\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2dx=\int_0^1\ln^2\left(\frac{x}{x+1}\right)dx+2\int_0^1\ln\left(\frac{x}{x+1}\right)dx+\int_0^1dx=$$

$$\int_0^1\ln^2(x)dx-2\int_0^1\ln(x+1)\ln(x)\,dx+\int_0^1\ln^2(x+1)dx$$

$$+2\int_0^1\ln(x)\,dx-2\int_0^1\ln(x+1)dx$$

$$+\int_0^1dx=2+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1x^n\ln(x)\,dx+\int_1^2\ln^2(x)dx-2$$

$$+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1x^ndx+1$$

$$B=2+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{d}{dn}\left(\frac{x^{n+1}}{n+1}\Bigg|_0^1\right)+2(\ln(2)-1)^2-2+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{x^{n+1}}{n+1}\Bigg|_0^1\right)+1$$

$$B=1-4+\frac{\pi^2}{6}+4\ln(2)+2\ln^2(2)-4\ln(2)+2+2-4\ln(2)$$

$$B=1+\frac{\pi^2}{6}-4\ln(2)+2\ln^2(2)\quad This:\ I=A+B$$

$$\int_0^1\left(x\ln\left(\cos^{-1}(1-x^2)\right)+\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2\right)dx$$

$$=\frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right)-\gamma\right]+2\ln^2(2)-4\ln(2)+1+\frac{\pi^2}{6}$$

Note : $''Ci(z)''$ *Cosine integral* … $Ci(z)=\ln(z)+\gamma+\int_0^z\frac{\cos(x)-1}{x}dx$

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Solution 3 by Exodo Halcalias-Angola

$$\int_0^1 \left(x ln\left(cos^{-1}(1-x^2)\right) + \left(\ln\left(\frac{x}{x+1}\right)+1\right)^2\right)dx$$

$$H_1 = \int_0^1 x\, ln\left(cos^{-1}(1-x^2)\right)dx \overset{1-x^2\to x}{=} \frac{1}{2}\int_0^1 ln\left(cos^{-1}(1-x^2)\right)dx \overset{cos^{-1}(x)\to x}{=} \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(x)\sin(x)\,dx$$

$$\overset{IBP}{=} \frac{1}{2}\left[(\ln(x)\,(1-\cos(x)))\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos(x)-1}{x}dx\right] = \frac{1}{2}\left[\ln\left(\frac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} \frac{\cos(x)-1}{x}dx\right] =$$

$$\frac{1}{2}\left(\mathrm{Ci}\left(\frac{\pi}{2}\right)+\gamma\right) \qquad H_1 = \frac{1}{2}\left(\mathrm{Ci}\left(\frac{\pi}{2}\right)+\gamma\right)$$

$$H_2 = \int_0^1 \left(\ln\left(\frac{x}{1+x}\right)+1\right)^2 dx = \int_0^1 ln^2\left(\frac{x}{1+x}\right)dx + 2\int_0^1 \ln\left(\frac{x}{x+1}\right)dx + \int_0^1 dx$$

$$A = \int_0^1 ln^2\left(\frac{x}{1+x}\right)dx = \int_0^1 ln^2(x)dx + \int_0^1 ln^2(1+x)dx - 2\int_0^1 \ln(x)\ln(x+1)\,dx =$$

$$A_1 = \int_0^1 ln^2(x)dx + \int_0^1 ln^2(1+x)dx = \int_0^1 ln^2(x)dx + \int_1^2 ln^2(x)dx$$

$$\int ln^2(z)dz = z(ln^2(z) - 2\ln(z) - 2) \qquad A_1 = 2ln^2(2) - 4\ln(2) + 4$$

$$A_2 = \int_0^1 \ln(x)\ln(1+x)\,dx \overset{IBP}{=} [x ln(1+x)(\ln(x)-1)]_0^1 - \int_0^1 \frac{x(\ln(x)-1)}{1+x}dx$$

$$A_2 = -\ln(2) + \int_0^1 \frac{x}{1+x}dx - \int_0^1 \frac{x ln(x)}{1+x}dx = -\ln(2) + \int_0^1 \frac{1+x-1}{1+x}dx -$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^k\ln(x)\,dx = -\ln(2) + 1 - \ln(2) + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+1)^2} = -2\ln(2) + 2 - \frac{\pi^2}{12}$$

$$A = A_1 - 2A_2 = \frac{\pi^2}{6} + 2ln^2(2)$$

$$B = \int_0^1 \ln\left(\frac{x}{1+x}\right)dx = \int_0^1 \ln(x)\,dx - \int_0^1 \ln(1+x)\,dx = \int_0^1 \ln(x)\,dx - \int_1^2 \ln(x)\,dx = -2\ln(2)$$

$$H_2 = A + 2B + 1 = \frac{\pi^2}{6} + 2ln^2(2) - 4\ln(2) + 1$$

$$H = H_1 + H_2 = \frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right) - \gamma\right] + 2ln^2(2) - 4\ln(2) + 1 + \frac{\pi^2}{6}$$