ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^\infty \frac{\ln(1+x^3)(\ln(1+y)+tan^{-1}(y))}{(1+y)^2(1+x^2)}dxdy$$

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$$We\ have:\int_0^1\int_0^\infty \frac{\ln(1+x^3)(\ln(1+y)+tan^{-1}(y))}{(1+y)^2(1+x^2)}dxdy\to$$

$$\underbrace{\int_0^\infty \frac{\ln(1+x^3)}{1+x^2}dx}_{\Omega_1}\underbrace{\int_0^1 \frac{\ln(1+y)+tan^{-1}(y)}{(1+y)^2}dy}_{\Omega_2}\quad (1)$$

$$Consider,\qquad \Omega(a)=\int_0^\infty \frac{\ln(a+x)}{1+x^2}dx\quad (2)$$

Differentiate above equation both sides with respect to 'a'. We get,

$$\overset{Leibniz\ Integral\ (rule)}{\Longrightarrow}\quad \frac{d}{da}\Omega(a)=\int_0^\infty \frac{\partial}{\partial a}\left(\frac{\ln(a+x)}{1+x^2}\right)dx\to\int_0^\infty \underbrace{\frac{dx}{(a+x)(1+x^2)}}_{Partical\ fraction}$$

$$\to\frac{a}{1+a^2}\int_0^\infty \frac{dx}{1+x^2}+\frac{1}{1+a^2}\int_0^\infty\left(\frac{1}{a+x}-\frac{1}{1+x^2}\right)dx\to\frac{a}{1+a^2}\int_0^\infty darctan(x)$$

$$+\frac{a}{1+a^2}\int_0^\infty dln\left(\frac{a+x}{\sqrt{1+x^2}}\right)\to\frac{d}{da}\Omega(a)=\frac{\pi a}{2(1+a^2)}-\frac{\ln(a)}{1+a^2}$$

Integrate above equation both sides with respect to. We get

$$\to\int\frac{d}{da}\Omega(a)\ da=\frac{\pi}{2}\int\frac{a}{1+a^2}da-\underbrace{\int\frac{\ln(a)}{1+a^2}da}_{I.B.P}\to\Omega(a)+C=$$

$$\frac{\pi}{4}\int dln(1+a^2)-\ln(a)\arctan(a)+\Im\underbrace{\int\frac{\ln(1+ia)}{a}}_{=-Li_2(ia)}=\Omega(a)+C=$$

$$\frac{\pi}{4}\ln(1+a^2)-\ln(a)\arctan(a)-\Im\{Li_2(-ia\}\underbrace{\to}_{Set\ a=0}\Omega(a=0)+C=-C$$

$$\int_0^\infty\frac{\ln(x)}{1+x^2}dx=C\to\int_0^\infty\frac{\ln(x)}{1+x^2}dx-\int_0^\infty\frac{\ln(x)}{1+x^2}dx=-2C\to C=0$$

$$\Omega(a)=\int_0^\infty\frac{\ln(a+x)}{1+x^2}dx=\frac{\pi}{4}\ln(1+a^2)-\ln(a)\arctan(a)-\Im\{Li_2(-ia\}\quad (3)$$

$$Now,\Omega_1=\int_0^\infty\frac{\ln(1+x^3)}{1+x^2}dx\to\int_0^\infty\frac{\ln(1+x)}{1+x^2}dx+\int_0^\infty\ln\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\frac{dx}{1+x^2}$$

$$+\int_0^\infty\ln\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\frac{dx}{1+x^2}\overset{Utilising\ \ Equation-(3)}{\Longrightarrow}\frac{\pi}{4}\ln(2)-\Im\{Li_2(-i)\}+\frac{\pi}{4}\ln\left(\frac{1+i\sqrt{3}}{2}\right)-$$

$$-\ln\left(\frac{-1-i\sqrt{3}}{2}\right)arctan\left(\frac{-1-i\sqrt{3}}{2}\right)-\Im\left\{Li_2\left(\frac{i-\sqrt{3}}{2}\right)\right\}+\frac{\pi}{4}\ln\left(\frac{1+i\sqrt{3}}{2}\right)-$$

$$\ln\left(\frac{-1+i\sqrt{3}}{2}\right)arctan\left(\frac{-1+i\sqrt{3}}{2}\right)-\Im\left\{Li_2\left(\frac{i+\sqrt{3}}{2}\right)\right\}$$

$$\left(\ln(x\pm iy)=\frac{\ln(x^2+y^2)}{2}\pm iarctan\left(\frac{y}{x}\right), x>0 \text{ and } y>0\right)$$

$$\left(Li_s(\pm i)=-2^{-s}\eta(s)\pm i\beta(s)\right)$$

$$\left(Cl_{2m}(\theta)=\Im\{Li_{2m}(e^{i\theta})\}\ m\geq 1\right)$$

$$\frac{\pi}{4}\ln(2)+\beta(2)-\frac{2i\pi}{3}\left(\frac{\pi}{4}+i\frac{\ln(2+\sqrt{3})}{2}\right)-\frac{2i\pi}{3}\left(-\frac{\pi}{4}+i\frac{\ln(2+\sqrt{3})}{2}\right)-Cl_2\left(\frac{\pi}{6}\right)-Cl_2\left(\frac{5\pi}{6}\right)$$

$$\overset{\beta(2)=G}{\Longrightarrow}-2\sum_{n\in N}\frac{sin\left(\frac{\pi n}{2}\right)cos\left(\frac{\pi n}{3}\right)}{n^2}+G+\frac{\pi}{4}\ln(2)-\frac{i\pi^2}{6}+\frac{\pi}{3}\ln(2+\sqrt{3})+\frac{i\pi^2}{6}+\frac{\pi}{3}\ln(2+\sqrt{3})=$$

$$-2\sum_{n\in Z_0^+}\frac{(-1)^n}{(6n+3)^2}-\sum_{n\in Z_0^+}\frac{(-1)^n}{(2n+1)^2}-\sum_{n\in N}\frac{(-1)^{n-1}}{(6n-3)^2}+G+\frac{2\pi}{3}\ln(2+\sqrt{3})+\frac{\pi}{4}\ln(2)\longrightarrow$$

$$-\frac{2G}{9}-G-\frac{G}{9}+G+\frac{2\pi}{3}\ln(2+\sqrt{3})+\frac{\pi}{4}\ln(2)$$

$$Now, \Omega_2=\int_0^1\frac{\ln(1+y)+tan^{-1}(y)}{(1+y)^2}dy\overset{I.B.P}{\cong}-\left(\frac{\ln(1+y)+\arctan(y)}{1+y}\right)\Big|_0^1+\int_0^1\frac{dy}{(1+y)^2}+$$

$$+\int_0^1\underbrace{\frac{dy}{(1+y)(1+y^2)}}_{Partial\ fraction}=-\frac{1}{2}\left(\frac{\pi}{4}+\ln(2)\right)-\int_0^1 d\left(\frac{1}{1+y}\right)-\frac{1}{2}\int_0^1\frac{y}{1+y^2}dy+\frac{1}{2}\int_0^1\frac{dy}{1+y}+$$

$$+\frac{1}{2}\int_0^1\frac{1}{1+y^2}dy=-\frac{\pi}{8}-\frac{\ln(2)}{2}-\frac{\ln(2)}{4}+\frac{\ln(2)}{2}+\frac{1}{2}+\frac{\pi}{8}$$

$$\Omega_2=\int_0^1\frac{\ln(1+y)+tan^{-1}(y)}{(1+y)^2}dy=\frac{2-\ln(2)}{4}$$

Put the value of Ω_1 ***and*** Ω_2 ***in equation*** $-(1)$. ***We get :***

$$\Omega_1\ .\ \Omega_2=\int_0^1\int_0^\infty\frac{\ln(1+x^3)(\ln(1+y)+tan^{-1}(y))}{(1+y)^2(1+x^2)}dxdy=$$

$$\left(\frac{2-\ln(2)}{4}\right)\left(-\frac{G}{3}+\frac{2\pi}{3}\ln(2+\sqrt{3})+\frac{\pi}{4}\ln(2)\right)$$

Note : $G\rightarrow$ Catalan's constant