ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\infty}\frac{x}{\sinh(x)}dx-\int_0^{\infty}\frac{y}{\sinh(3y)}dy+\int_0^{\infty}\frac{z}{\sinh(5z)}dz-\cdots=\frac{\pi^2}{4}G$$

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$$\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}\frac{x}{\sinh(x(2n+1))}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}\frac{x}{\frac{e^{x(2n+1)}-e^{-x(2n+1)}}{2}}dx=$$

$$=\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}\frac{2x}{e^{x(2n+1)}-e^{-x(2n+1)}}dx=2\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}\frac{x}{e^{x(2n+1)}-e^{-x(2n+1)}}dx=$$

$$=-2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^1\frac{\ln(t)}{1-t^2}dt=-2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sum_{m=0}^{\infty}\int_0^1 t^{2m}\ln(t)\,dt=$$

$$2G\left(\zeta(2)-\frac{1}{4}-\frac{1}{16}-\cdots\right)=2G\left(\zeta(2)-\frac{\zeta(2)}{4}\right)=\frac{\pi^2}{4}G$$

Notes :

1) $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}=\zeta(2)$

2) $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G$

3) $\sinh(x)=\frac{e^x-e^{-x}}{2}$

4) $\sinh\big(x(2n+1)\big)=\frac{e^{x(2n+1)}-e^{-x(2n+1)}}{2}$