ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin(y)}{\sin(x)}\sqrt{\frac{\sin(2x)}{\sin(2y)}}\,dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin(y)}{\sin(x)}\sqrt{\frac{\sin(2x)}{\sin(2y)}}\,dxdy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(2x)}{sin^2(x)}}\cdot\sqrt{\frac{sin^2(y)}{\sin(2y)}}\,dxdy=$$

$$\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(2x)}{sin^2(x)}}\,dx\int_0^{\frac{\pi}{2}}\sqrt{\frac{sin^2(y)}{\sin(2y)}}\,dy=\int_0^{\frac{\pi}{2}}\sqrt{\frac{2\sin(x)\cos(x)}{\sin(x)\,.\sin(x)}}\,dx\,.\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(y)\,.\sin(y)}{2\sin(y)\cos(y)}}\,dy$$

$$\int_0^{\frac{\pi}{2}}\sqrt{\frac{2\cos(x)}{\sin(x)}}\,dx.\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(y)}{2\cos(y)}}\,dy=\int_0^{\frac{\pi}{2}}\sqrt{2}\cos(x)^{\frac{1}{2}}\sin(x)^{-\frac{1}{2}}dx.\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2}}\sin(y)^{\frac{1}{2}}\cos(y)^{-\frac{1}{2}}dy=$$

$$\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2},\frac{-\frac{1}{2}+1}{2}\right).\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2},\frac{\frac{1}{2}+1}{2}\right)=\frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{4}\right).\beta\left(\frac{1}{4},\frac{3}{4}\right)=$$

$$\frac{1}{4}\Gamma\left(\frac{1}{4},\frac{3}{4}\right).\Gamma\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{4}.\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}.\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}=\frac{1}{4}.\frac{4\pi^2}{2}=\frac{\pi^2}{2}$$

Note :

$$\beta(m,n)=\frac{\Gamma(m)\,.\,\Gamma(n)}{\Gamma(m+n)}\quad,\quad\Gamma(n)\,.\,\Gamma(1-n)=\frac{\pi}{\sin(n\pi)}$$