ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1\int_0^1 \frac{\ln(1-x^2y^2)-xy\ln\left(\frac{1-xy}{1+xy}\right)}{1-x^2y^2}dxdy$$

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Solution by Quadri Faruk Temitope-Nigeria

$$\int_0^1\int_0^1 \frac{\ln(1-x^2y^2)-xy\ln\left(\frac{1-xy}{1+xy}\right)}{1-x^2y^2}dxdy =$$

$$I=\int_0^1\int_0^1 \frac{\ln(1-x^2y^2)}{1-x^2y^2}dxdy-\int_0^1\int_0^1 \frac{xy\ln\left(\frac{1-xy}{1+xy}\right)}{1-x^2y^2}dxdy=$$

$$\int_0^1\int_0^1 \frac{\ln(1-xy)+\ln(1+xy)}{1-x^2y^2}dxdy-\int_0^1\int_0^1 \frac{xy(\ln(1-xy)-\ln(1+xy))}{1-x^2y^2}dxdy=$$

$$\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(1-xy)}{1+xy}dxdy\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(1-xy)}{1-xy}dxdy+\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(1+xy)}{1+xy}dxdy+$$

$$+\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(1+xy)}{1-xy}dxdy+\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(1-xy)}{1+xy}dxdy\frac{-1}{2}\int_0^1\int_0^1 \frac{\ln(1-xy)}{1-xy}dxdy-$$

$$\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(1+xy)}{1+xy}dxdy+\frac{1}{2}\int_0^1\int_0^1 \frac{\ln(1+xy)}{1-xy}dxdy$$

$$I=\int_0^1\int_0^1 \frac{\ln(1-xy)}{1+xy}dxdy+\int_0^1\int_0^1 \frac{\ln(1+xy)}{1-xy}dxdy$$

$$I=\int_0^1\left[\frac{\ln(1-xy)\ln(1+xy)}{x}\Bigg|_0^1+\int_0^1 \frac{\ln(1+xy)}{1-xy}dy\right]dx+\int_0^1\int_0^1 \frac{\ln(1+xy)}{1-xy}dxdy$$

$$I=\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx\ +\ 2\int_0^1\int_0^1 \frac{\ln(1+xy)}{1-xy}dxdy$$

***Recall that*:**

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx=-\frac{5}{8}\zeta(3)$$

***Hence*:**

$$\int_0^1\int_0^1 \frac{\ln(1+xy)}{1-xy}dxdy=\frac{\pi^2}{4}\ln(2)-\zeta(3)$$

$$I=-\frac{5}{8}\zeta(3)+2\left[\frac{\pi^2}{4}\ln(2)-\zeta(3)\right]=\frac{\pi^2}{2}\ln(2)-\frac{21}{8}\zeta(3)$$