ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \left((\ln^4(x) + x^4) + \frac{x^3}{3 - x^3} \right) dx$$

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$$I = \int_0^1 \left((\ln^4(x) + x^4) \ + \frac{x^3}{3 - x^3} \right) dx = \underbrace{\int_0^1 \ln^4(x)dx}_{A} + \underbrace{\int_0^1 x^4 dx}_{B} + \underbrace{\int_0^1 \frac{x^3}{3 - x^3} dx}_{C}$$

$$A = \int_0^1 \ln^4(x)dx \underbrace{\overset{\substack{\ln(x)=-p \\ x=e^{-p}}}{=}}_{\substack{dx=-e^{p}dp \\ [-\infty;0]}} \int_{-\infty}^{0} (-p)^4.(-e^{-p})dp = \int_0^{+\infty} p^{5-1}e^{-p} = \Gamma(5) = 24$$

$$B = \int_0^1 x^4 dx = \frac{1}{5}x^5 \begin{matrix} 1 \\ 0 \end{matrix} = \frac{1}{5}$$

$$C = \int_0^1 \frac{x^3}{3 - x^3} dx = \frac{1}{3}\int_0^1 \frac{x^3}{1 - (\frac{x}{\sqrt{3}})^3} dx = \frac{1}{3}\sum_{n=0}^{\infty}\int_0^1 (\frac{x}{\sqrt{3}})^{3n}.x^3\, dx =$$

$$\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{(\sqrt{3})^{3n}}\int_0^1 x^{3n+3}\, dx = \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}\cdot\left(\frac{x^{3n+4}}{3n+4}\right)\begin{matrix} 1 \\ 0 \end{matrix} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}\cdot\frac{1}{(3n+4)} =$$

$$= -\frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)}$$

$$\textit{Hence}:$$

$$I = A + B + C = 24 + \frac{1}{5} - \frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)} = \frac{121}{5} - \frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)}$$

$$\int_0^1 \left((\ln^4(x) + x^4) + \frac{x^3}{3 - x^3} \right) dx = \frac{121}{5} - \frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)}$$