ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I = \int_{0}^{1} x(\log(\log x) + \arctan(x-1)) dx = \frac{1}{2}(i\pi - \gamma - 1)$$

γ is the *Euler-Mascheroni constant*.

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$\begin{split} I &= \int\limits_{0}^{1} x (log(logx) + arctan(x-1)) \, dx = \int\limits_{0}^{1} x log(log(x)) \, dx + \int\limits_{0}^{1} x arctan(x-1) \, dx \\ I_{1} &= \int\limits_{0}^{1} x log(log(x)) \, dx \bigg|_{log(x) = -y} &= \int\limits_{0}^{\infty} e^{-2y} log(-y) \, dy \\ &= log(-1) \int\limits_{0}^{\infty} e^{-2y} \, dy + \int\limits_{0}^{\infty} e^{-2y} log(y) \, dy \bigg|_{2y = z} \\ &= \frac{i\pi}{2} + \frac{1}{2} \int\limits_{0}^{\infty} e^{-z} log(z) \, dz - \frac{log(2)}{2} \int\limits_{0}^{\infty} e^{-z} \, dz = \frac{i\pi}{2} - \frac{1}{2} \gamma - \frac{log(2)}{2} \\ &= \frac{1}{2} (i\pi - \gamma - log(2)) \\ I_{2} &= \int\limits_{0}^{1} x arctan(x-1) \, dx = -\int\limits_{0}^{1} (1-x) arctan(x) \, dx = \frac{1}{2} (log(2)-1) \\ &I = I_{1} + I_{2} = \frac{1}{2} (i\pi - \gamma - 1) \\ &Note: \int\limits_{0}^{\infty} e^{-z} log(z) \, dz = -\gamma \end{split}$$