ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^1 x(\log(\log x)+\arctan(x-1))\,dx=\frac{1}{2}(i\pi-\gamma-1)$$

γ is the *Euler-Mascheroni constant.*

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1 x(\log(\log x)+\arctan(x-1))\,dx=\int_0^1 x\log(\log(x))\,dx+\int_0^1 x\arctan(x-1)\,dx$$

$$I_1=\int_0^1 x\log(\log(x))\,dx\Bigg]_{\log(x)=-y}=\int_0^\infty e^{-2y}\log(-y)\,dy$$

$$=\log(-1)\int_0^\infty e^{-2y}\,dy+\int_0^\infty e^{-2y}\log(y)\,dy\Bigg]_{2y=z}$$

$$=\frac{i\pi}{2}+\frac{1}{2}\int_0^\infty e^{-z}\log(z)\,dz-\frac{\log(2)}{2}\int_0^\infty e^{-z}\,dz=\frac{i\pi}{2}-\frac{1}{2}\gamma-\frac{\log(2)}{2}$$

$$=\frac{1}{2}(i\pi-\gamma-\log(2))$$

$$I_2=\int_0^1 x\arctan(x-1)\,dx=-\int_0^1(1-x)\arctan(x)\,dx=\frac{1}{2}(\log(2)-1)$$

$$I=I_1+I_2=\frac{1}{2}(i\pi-\gamma-1)$$

$$\text{Note: }\int_0^\infty e^{-z}\log(z)\,dz=-\gamma$$