ROMANIAN MATHEMATICAL MAGAZINE

$$\Omega_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\frac{1}{\cos(x)} + \frac{1}{\sin(x)}\right)dx \ \ and$$

$$\Omega_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\frac{1}{\cos(y)} - \frac{1}{\sin(y)}\right)dy$$

$$Prove\ \ that:\ \Omega_1 + \Omega_2 = \frac{21}{64}\zeta(3) + \frac{9}{32}\pi^2\ln(2)$$

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$$N = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\frac{1}{\cos(x)} + \frac{1}{\sin(x)}\right)dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\frac{\sin(x)+\cos(x)}{\sin(x)\cos(x)}\right)dx =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)+\cos(x))dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)\cos(x))dx =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)+\cos(x))dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x))dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\cos(x))dx \ \ = A_1 + A_2 + A_3$$

This:

$$A_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)+\cos(x))dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\sqrt{2}\sin\left(\frac{3}{4}\pi - x\right)\right)dx =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sqrt{2})\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin\left(\frac{3}{4}\pi - x\right))dx = \frac{1}{2}\ln(2)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} xdx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{3}{4}\pi - x\right)\ln(sin(x))\,dx =$$

$$\frac{x^2}{4}\ln(2) + \frac{3}{4}\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin(x))\,dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \frac{1}{4}\ln(2)\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) + \frac{3}{4}\pi\left(\frac{G}{2} - \frac{\pi}{4}\ln(2)\right) -$$

$$\left(\frac{\pi}{8}G + \frac{21}{128}\zeta(3) - \frac{3}{32}\pi^2\ln(2)\right),$$

$$A_1 = \frac{\pi}{4}G - \frac{21}{128}\zeta(3) - \frac{3}{64}\pi^2\ln(2)$$

$$A_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x))dx = \frac{\pi}{8}G + \frac{21}{128}\zeta(3) - \frac{3}{32}\pi^2\ln(2)$$

$$A_3 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\cos(x))dx \ = -\frac{\pi}{8}G - \frac{3}{32}\pi^2\ln(2) - \frac{35}{128}\zeta(3)$$

$$I = A_1 - A_2 - A_3 = \left(\frac{\pi}{4}G - \frac{21}{128}\zeta(3) - \frac{3}{64}\pi^2\ln(2)\right) - \left(\frac{\pi}{8}G + \frac{21}{128}\zeta(3) - \frac{3}{32}\pi^2\ln(2)\right) -$$

$$\left(-\frac{\pi}{8}G - \frac{3}{32}\pi^2\ln(2) - \frac{35}{128}\zeta(3)\right) = \frac{\pi}{4}G - \frac{7}{128}\zeta(3) + \frac{9}{64}\pi^2\ln(2)$$

Hence:

$$M=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} yln\left(\frac{1}{\cos(y)}-\frac{1}{\sin(y)}\right)dy=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} yln\left(\frac{\sin(y)-\cos(y)}{\sin(y)\cos(y)}\right)dy=$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} yln(\sin(y)-\cos(y))dy-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} yln(\sin(y)\cos(y))dy=B_1-B_2-B_3$$

Since the $M-$integral is symmetric with the $N-$integral, we can directly write the answer of the $M-$integral, the integral's are similar …

$$B_1=\frac{1}{4}\ln(2)\left(\frac{\pi^2}{4}-\frac{\pi^2}{16}\right)+\frac{3}{4}\pi\left(-\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\left(-\frac{\pi}{8}G-\frac{35}{128}\zeta(3)-\frac{3}{32}\pi^2\ln(2)\right)$$

$$B_1=-\frac{\pi}{4}G+\frac{35}{128}\zeta(3)-\frac{3}{64}\pi^2\ln(2)\ ,\ \ M=B_1-B_2-B_3$$

$$M=\left(-\frac{3}{64}\pi^2\ln(2)-\frac{\pi}{4}G+\frac{35}{128}\zeta(3)\right)-\left(\frac{\pi}{8}G+\frac{21}{128}\zeta(3)-\frac{3}{32}\pi^2\ln(2)\right)-$$

$$\left(-\frac{\pi}{8}G-\frac{35}{128}\zeta(3)-\frac{3}{32}\pi^2\ln(2)\right)=\frac{49}{128}\zeta(3)-\frac{\pi}{4}G+\frac{9}{64}\pi^2\ln(2)$$

$$\textit{Then}:\Omega_1+\Omega_2=N+M$$

$$\Omega_1+\Omega_2=\frac{\pi}{4}G-\frac{7}{128}\zeta(3)+\frac{9}{64}\pi^2\ln(2)+\frac{49}{128}\zeta(3)-\frac{\pi}{4}G+\frac{9}{64}\pi^2\ln(2)$$

$$\Omega_1+\Omega_2=\frac{21}{64}\zeta(3)+\frac{9}{32}\pi^2\ln(2)$$