ROMANIAN MATHEMATICAL MAGAZINE

**Prove that:**

$$\Omega=\int_0^\infty\int_0^1\frac{x\ln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2}dxdy=-\frac{1}{48}(\pi^2-9\zeta(3))(4G+\pi(\ln(2)-1)$$

***Proposed by Shirvan Tahirov-Azerbaijan***

***Solution 1 by Exodo Halcalias-Angola***

$$\int_0^\infty\int_0^1\frac{x\ln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2}dxdy=\int_0^1\frac{x\ln^2(x)}{(x+1)^2}dx\int_0^\infty\frac{\ln(y+1)}{(1+y^2)^2}dy=$$

$$\left(\int_0^1\frac{\ln^2(x)}{x+1}dx-\int_0^1\frac{\ln^2(x)}{(x+1)^2}dx\right)\int_0^\infty\frac{\ln(y+1)}{(1+y^2)^2}dy=$$

$$\left(\sum_{k\in N}(-1)^{k-1}\int_0^1x^{k-1}\ln^2(x)dx+\sum_{n\in N}(-1)^n n\int_0^1x^{n-1}\ln^2(x)dx\right)$$

$$\cdot\left(\int_1^\infty+\int_0^1\right)\frac{\ln(y+1)}{(1+y^2)^2}dy=2\left(\sum_{k\in N}\frac{(-1)^{k-1}}{k^3}-\sum_{k\in N}\frac{(-1)^{n-1}}{n^2}\right)\left(\int_1^\infty\frac{\ln(y+1)}{(1+y^2)^2}dy+\right.$$

$$\left.\int_0^1\frac{\ln(y+1)}{(1+y^2)^2}dy\right)=2\big(\eta(3)-\eta(2)\big)\left(\int_0^1\frac{\ln\left(\frac{1}{y}+1\right)}{\left(\frac{1}{y^2}+1\right)^2y^2}dy+\int_0^1\frac{\ln(y+1)}{(1+y^2)^2}dy\right)=$$

$$2\Big((1-2^{1-3})\zeta(3)-(1-2^{1-2})\zeta(2)\Big)\left(\int_0^1\frac{y^2\ln(y+1)-\ln(y)}{(y^2+1)}dy+\int_0^1\frac{\ln(y+1)}{(1+y^2)^2}dy\right)=$$

$$\left(\frac{3\zeta(3)}{2}-\zeta(2)\right)\left(\int_0^1\frac{\ln(y+1)}{1+y^2}dy-\int_0^1\frac{\ln(y+1)}{(1+y^2)^2}dy-\int_0^1\frac{y^2\ln(y)}{(1+y^2)^2}dy+\int_0^1\frac{\ln(y+1)}{(1+y^2)^2}dy\right)=$$

$$\left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)\left(\int_0^1\frac{\ln(y+1)}{1+y^2}dy-\int_0^1\frac{y^2\ln(y)}{(1+y^2)^2}dy\right)=\left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)\left(\int_0^1\frac{\ln\left(\frac{2}{y+1}\right)}{y^2+1}dy-\right.$$

$$\left.-\frac{1}{2}\left(-\frac{y\ln(y)}{y^2+1}+\ln(y)\tan^{-1}(y)\right)\Bigg|_0^1+\frac{1}{2}\int_0^1\left(-\frac{1}{1+y^2}+\frac{\tan^{-1}(y)}{y}\right)dy\right)=\left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)$$

$$\left(\frac{\ln(2)}{2}\int_0^1\frac{dy}{y^2+1}-\frac{1}{2}\int_0^1\frac{dy}{y^2+1}+\frac{1}{2}\int_0^1\frac{\tan^{-1}(y)}{y}dy\right)=\left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)\left(\frac{\ln(2)}{2}\cdot\frac{\pi}{4}-\frac{1}{2}\cdot\frac{\pi}{4}+\right.$$

$$+\frac{Ti_2(1)}{2}\Big) = \left(\frac{3\zeta(3)}{2} - \frac{\pi^2}{6}\right)\left(\pi\frac{\ln(2)}{8} - \frac{\pi}{8} + \frac{G}{2}\right)$$

$$\int_0^\infty \int_0^1 \frac{x\ln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2} dxdy = -\frac{1}{48}(\pi^2 - 9\zeta(3))(4G + \pi(\ln(2) - 1)$$

*Solution 2 by Quadri Faruk Temitope-Nigeria*

$$I = \int_0^\infty \int_0^1 \frac{x\ln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2} dxdy = \int_0^1 \frac{x\ln^2(x)}{(x+1)^2} dx \int_0^\infty \frac{\ln(y+1)}{(1+y^2)^2} dy = A\ .\ B$$

$working\ or\ A:$

$$A = \int_0^1 \frac{x\ln^2(x)}{(x+1)^2} dx = -\sum_{n=1}^\infty (-1)^n n \int_0^1 x^n \ln^2(x) dx = -\sum_{n=1}^\infty (-1)^n n \frac{d^2}{dn^2}(\int_0^1 x^n dx) =$$

$$-\sum_{n=1}^\infty (-1)^n n\left(\frac{2}{(1+n)^3}\right) = -2\sum_{n=1}^\infty (-1)^n \left(\frac{n}{(1+n)^3}\right) = -2\sum_{n=1}^\infty (-1)^n \left(\frac{1}{(1+n)^2} - \frac{1}{(1+n)^3}\right) =$$

$$-2\sum_{n=1}^\infty \frac{(-1)^n}{(1+n)^2} + 2\sum_{n=1}^\infty \frac{(-1)^n}{(1+n)^3} = -2\left(\frac{\pi^2}{12} - 1\right) + 2\left(\frac{3}{4}\zeta(3) - 1\right) = -\frac{\pi^2}{6} + 2 + \frac{3}{2}\zeta(3) - 2 =$$

$$= \frac{3}{2}\zeta(3) - \frac{\pi^2}{6} = \frac{1}{6}(9\zeta(3) - \pi^2)$$

$$A = \int_0^1 \frac{x\ln^2(x)}{(x+1)^2} dx = \frac{1}{6}(9\zeta(3) - \pi^2)$$

$working\ or\ B:$

$$B = \int_0^\infty \frac{\ln(y+1)}{(1+y^2)^2} dy \quad Recall\ that: \ln(y+1) = \int_0^1 \frac{y}{1+xy} dy$$

$$\int_0^1 \int_0^\infty \frac{y}{(1+xy)(1+y^2)^2} dxdy = \int_0^\infty \int_0^1 \frac{x^2 y}{(1+x^2)^2(y^2+1)} dxdy -$$

$$-\int_0^\infty \int_0^1 \frac{x}{(1+x^2)^2(y^2+1)} dxdy + \int_0^\infty \int_0^1 \frac{x}{(1+y^2)^2(x^2+1)} dxdy +$$

$$+\int_0^\infty \int_0^1 \frac{y}{(1+y^2)^2(x^2+1)} dxdy - \int_0^\infty \int_0^1 \frac{x^3}{(1+x^2)^2(xy+1)} dxdy$$

$$B = \int_0^\infty \frac{x^2}{(1+x^2)^2} dx \int_0^1 \frac{y}{y^2+1} dy - \int_0^\infty \frac{dy}{y^2+1} \int_0^1 \frac{x}{(1+x^2)^2} dx +$$

$$+\int_0^{\infty}\frac{dy}{(1+y^2)^2}\int_0^1\frac{x}{1+x^2}dx+\int_0^{\infty}\frac{dx}{1+x^2}\int_0^1\frac{y}{(1+y^2)^2}dy-\int_0^{\infty}\int_0^1\frac{x^3}{(1+x^2)^2(xy+1)}dxdy$$

$$B=\left[\frac{1}{2}tan^{-1}(x)-\frac{x}{2(x^2+1)}\right]_0^{\infty}.\frac{1}{2}\ln\left(y^2+1\right)_0^1-tan^{-1}(y)_0^{\infty}.\frac{(-1)}{2(x^2+1)}{}_0^1+$$

$$+\frac{1}{2}\left[\frac{y}{y^2+1}+tan^{-1}(y)\right]_0^{\infty}-\frac{1}{2}\ln\left(x^2+1\right)_0^1+tan^{-1}(x)_0^{\infty}.\frac{(-1)}{2(y^2+1)}{}_0^1-$$

$$-\int_0^{\infty}\int_0^1\frac{x^3}{(1+x^2)^2(xy+1)}dxdy$$

$$B=\frac{\pi}{4}\left(\frac{\ln(2)}{2}\right)-\frac{\pi}{2}.\frac{1}{4}+\frac{\pi}{4}\left(\frac{\ln(2)}{2}\right)+\frac{\pi}{2}.\frac{1}{4}-\frac{\pi}{2}.\frac{1}{4}+\frac{G}{2}-\frac{\pi}{4}\left(\frac{\ln(2)}{2}\right)$$

$$B=\frac{\pi}{8}\ln(2)-\frac{\pi}{8}+\frac{G}{2}=\frac{1}{8}(4G+\pi(\ln(2)-1)$$

$$B=\int_0^{\infty}\frac{\ln(y+1)}{(1+y^2)^2}dy=\frac{1}{8}(4G+\pi(\ln(2)-1)$$

$$I=A\,.\,B=\int_0^1\frac{xln^2(x)}{(x+1)^2}dx\int_0^{\infty}\frac{\ln(y+1)}{(1+y^2)^2}dy=\left(\frac{1}{6}\left(9\zeta(3)-\pi^2\right)\right).\left(\frac{1}{8}(4G+\pi(\ln(2)-1)\right)$$

$$=-\frac{1}{48}(\pi^2-9\zeta(3))(4G+\pi(\ln(2)-1)$$

***Note* : *G* − *Catalan's constant* ,  $\zeta(3)$ − *Apery's constant***