ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\psi=\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\frac{1}{2}\left(\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{16}\right)+1\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\therefore Si(z)=\int_0^1\frac{\sin t}{t}dt;\ Ci(z)=-\int_z^\infty\frac{\cos t}{t}dt=\gamma+\ln(z)+\int_0^z\frac{\cos t-1}{t}dt$$

$$\psi=\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=2\int_0^1 x\ln(arcsin(1-x))dx-$$

$$-\int_0^1 x\ln(x^2+1)dx-2A-B$$

$$\therefore A=\int_0^1 x\ln(arcsin(1-x))dx\overset{1-x\to 1}{\cong}\int_0^1(1-x)\ln(arcsin(x))dx=$$

$$=\int_0^1\ln(arcsin(x))dx-\int_0^1 x\ln(arcsin(x))dx\overset{x=\sin t}{\cong}\underbrace{\int_0^1\ln(t)d(\sin t)}_{IBP}-$$

$$-\int_0^{\frac{\pi}{2}}\cos(t)\sin(t)\,ln(t)dt=sin(t)ln(t)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{\sin t}{t}dt-\underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2t)ln(t)dt}_{2t\to t}=$$

$$=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+\frac{1}{4}\int_0^\pi\ln\left(\frac{t}{2}\right)d(\cos(t))=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+$$

$$+\frac{1}{4}\left\{\underbrace{\int_0^\pi\ln(t)d(\cos(t)-1)}_{IBP}-ln(2)\underbrace{\int_0^\pi d(\cos(t))}_{=-2}\right\}=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+$$

$$+\frac{1}{4}\left\{cos(t)-1)ln(t)\Big|_0^\pi-\int_0^z\frac{\cos t-1}{t}dt+2\ln(2)\right\}=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+\frac{1}{4}(-2ln(\pi)-Ci(\pi)+\gamma$$

$$+\ln(\pi)+2\ln(2))=\frac{1}{4}\left\{\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{4}\right)\right\}$$

$$\therefore B=\int_0^1 x\ln(x^2+1)dx\overset{IBP}{\cong}\frac{(x^2+1)\ln(x^2+1)}{2}\Big|_0^1-\int_0^1\frac{(x^2+1)x}{(x^2+1)}dx=\ln(2)-\int_0^1 xdx=\ln(2)-\frac{1}{2}$$

$$Therefore{:}\ \psi=2A-B=2\left\{\frac{1}{4}\left\{\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{4}\right)\right\}\right\}-\left(\ln(2)-\frac{1}{2}\right)=$$

$$=\frac{1}{2}\left(1+\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{16}\right)\right)$$

Solution 2 by Ankush Kumar Parcha-India

$$We\ have, \int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx = 2\underbrace{\int_0^1 xlnsin^{-1}(1-x)dx}_{x\to 1-x} - \underbrace{\int_0^1 xln(1+x^2)dx}_{x^2\to x}$$

$$= 2\underbrace{\int_0^1 (1-x)lnsin^{-1}dx}_{x\to\sin(x)} - \frac{1}{2}\underbrace{\int_0^1 ln(1+x)dx}_{I.B.P} \overset{\sin(2x)=2\sin(x)\cos(x)}{\cong} 2\underbrace{\int_0^{\frac{\pi}{2}} ln(x)\cos(x)\,dx}_{I.B.P} -$$

$$\underbrace{\int_0^{\frac{\pi}{2}} ln(x)\sin(2x)\,dx}_{I.B.P} - (\frac{\ln(1+x)}{2}\int dx)\frac{1}{0} + \frac{1}{2}\int_0^1 \frac{x+1-1}{x+1}dx = (2\ln(x)\int dsin(x))\frac{\frac{\pi}{2}}{0} -$$

$$-2\int_0^{\frac{\pi}{2}} dSi(x) + (\frac{\ln(x)}{2}\int dcos(2x))\frac{\frac{\pi}{2}}{0} - \int_0^{\frac{\pi}{2}}\frac{dCi(x)}{2} - \frac{\ln(2)}{2} + \int_0^1 \frac{dx}{2} - \frac{1}{2}\int_0^1 \frac{dx}{1+x}$$

$$(\int_0^x \frac{1-\cos(t)}{t}dt = {}_\gamma + \ln(x) - Ci(x),\ |Arg(x)| < \pi)$$

$$= 2\ln\left(\frac{\pi}{2}\right) - 2Si\left(\frac{\pi}{2}\right) - \frac{1}{2}\ln\left(\frac{\pi}{2}\right) - \lim_{x\to 0}\frac{\ln(x)}{2} - \frac{Ci(\pi)}{2} + \frac{\gamma}{2} + \lim_{x\to 0}\frac{\ln(2x)}{2} - \frac{\ln(2)}{2} + \frac{1}{2} -$$

$$\int_0^1 \frac{dln(1+x)}{2} = \frac{\gamma}{2} - 2Si\left(\frac{\pi}{2}\right) - \frac{Ci(\pi)}{2} + \frac{3}{2}\ln\left(\frac{\pi}{2}\right) + \frac{\ln(2)}{2} - \ln(2) + \frac{1}{2} =$$

$$\frac{\gamma}{2} - 2Si\left(\frac{\pi}{2}\right) - \frac{Ci(\pi)}{2} + \ln\left(\frac{\pi\sqrt{\pi}}{2}\right) - \ln\left(\frac{2}{\sqrt{e}}\right)$$

$$, \int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx = \frac{\gamma}{2} - 2Si\left(\frac{\pi}{2}\right) - \frac{Ci(\pi)}{2} + \ln\left(\frac{\pi}{4}\sqrt{\pi e}\right)$$

$$\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right) = \psi$$

$$\psi = \frac{1}{2}\left(\gamma - Ci(\pi) - 4Si\left(\frac{\pi}{2}\right) + \ln\left(\frac{\pi^3}{16}\right) + 1\right)$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$I = \int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx \quad Recall\ That:\ \ln\left(\frac{A}{B}\right) = ln(A) - ln(B)$$

$$I = \int_0^1 \underbrace{x\ln\left(arcsin^2(1-x)\right)}_{A}dx - \int_0^1 \underbrace{x\ln(1+x^2)}_{B}dx$$

$$Working\ or\ A:$$

$$A = \int_0^1 x\ln\left(arcsin^2(1-x)\right)dx = 2\int_0^1 x\ln(arcsin(1-x))dx\ \ let:\ 1-x = x$$

$$A=2\int_0^1(1-x)\ln(arcsin(1-x))dx=2\int_0^1\ln(arcsin(x))dx-2\int_0^1 x\ln(arcsin(x))dx$$

Very integration by parts :

$$A=2[xln(\sin^{-1}x))-Si(\sin^{-1}(x))]_0^1-2[x(x\ln(\sin^{-1}(x))-Si(\sin^{-1}(x))]_0^1-$$

$$-\int_0^1 x^2\ln(\sin^{-1}(x))dx+\int_0^1 x\,Si\,(\sin^{-1}(x))dx$$

$$A=2\left[\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)\right]-2\left[ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)-\frac{\gamma}{4}+\frac{1}{4}ln\left(\frac{\pi}{4}\right)+\frac{Ci(\pi)}{4}+\sin\left(\frac{\pi}{2}\right)-\ln\left(\frac{\pi}{2}\right)\right]$$

$$A=2\ln\left(\frac{\pi}{2}\right)-2Si\left(\frac{\pi}{2}\right)+\frac{\gamma}{2}-\frac{1}{2}\ln\left(\frac{\pi}{4}\right)-\frac{Ci(\pi)}{2}$$

$$A=2\ln\left(\frac{\pi}{2}\right)-\frac{1}{2}ln\left(\frac{\pi}{4}\right)+\frac{\gamma}{2}-2\,\mathrm{Si}\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}$$

- ***Working or B*** :

$$B=\int_0^1 x\ln(1+x^2)\,dx=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{2n+1}dx=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$$

$$B=-\frac{1}{2}[-\ln(2)]+\frac{1}{2}[\ln(2)-1]=\frac{ln(2)}{2}+\frac{ln(2)}{2}-\frac{1}{2}=ln(2)-\frac{1}{2}$$

This , $I=A-B$

$$I=2ln\left(\frac{\pi}{2}\right)-\frac{1}{2}ln\left(\frac{\pi}{4}\right)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)-\left[\ln(2)-\frac{1}{2}\right]$$

$$I=2ln\left(\frac{\pi}{2}\right)-\frac{1}{2}ln\left(\frac{\pi}{4}\right)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)-\ln(2)+\frac{1}{2}$$

Recall that: $\ln A^B=B\ln A$

$$\ln(A)+\ln(B)=ln(AB)$$

$$ln(A)-ln(B)=ln\left(\frac{A}{B}\right)$$

$$I=ln\left(\frac{\pi^2}{4}\right)-ln\left(\frac{\sqrt{\pi}}{2}\right)-\ln(2)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)+\frac{1}{2}$$

$$I=ln\left(\frac{\pi^{3/2}}{4}\right)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)+\frac{1}{2}$$

$$I=\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\frac{1}{2}\left[ln\left(\frac{\pi^3}{16}\right)+\gamma-4Si\left(\frac{\pi}{2}\right)-Ci(\pi)+1\right]$$

Solution 4 by Exodo Halcalias-Angola

$$\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\underbrace{\int_0^1 xlnarcsin^2(1-x)dx}_{\arcsin(1-x)\to x}-\underbrace{\int_0^1 xln(x^2+1)dx}_{x^2+1\to y}=$$

$$\int_0^{\frac{\pi}{2}}(1-\sin(x))dxcos(x)\ln(x^2)dx-\frac{1}{2}\int_1^2 \ln(x)\,dx=2\underbrace{\int_0^{\frac{\pi}{2}}\cos(x)\ln(x)\,dx}_{I.B.P}-$$

$$2\int_0^{\frac{\pi}{2}}\sin(x)\cos(x)\ln(x)\,dx-\frac{1}{2}(yln(y)-y)\begin{matrix}2\\1\end{matrix}=2((\ln(x)\sin(x))\begin{matrix}\frac{\pi}{2}\\0\end{matrix}-\int_0^{\frac{\pi}{2}}\frac{\sin(x)}{x}dx)-$$

$$\int_0^{\frac{\pi}{2}}\ln(x)\sin(2x)\,dx\,[2x\to t]-\frac{1}{2}(2\ln(2)-2+1)=2(\ln\left(\frac{\pi}{2}\right)-sin\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right))-$$

$$\frac{1}{2}\int_0^{\pi}\sin(t)\ln\left(\frac{t}{2}\right)-\ln(2)+\frac{1}{2}=2(\left(\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)\right)-\frac{1}{2}\underbrace{\int_0^{\pi}\sin(t)\ln(t)\,dt}_{I.B.P}+$$

$$\frac{1}{2}\int_0^{\pi}\sin(t)\ln(2)\,dt+\ln\left(\frac{1}{2}\right)+\frac{1}{2}=2\ln\left(\frac{\pi}{2}\right)-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}(-\ln(t)\cos(t)\begin{matrix}\pi\\0\end{matrix}+\int_0^{\pi}\frac{cost}{t}dt)$$

$$-\frac{\ln(2)}{2}\int_0^{\pi}dcost+\ln\left(\frac{1}{2}\right)+\frac{1}{2}=2\ln(\frac{\pi}{2})-2Si(\frac{\pi}{2})-\frac{1}{2}(\ln(\pi)+\int_0^{\pi}\frac{dt}{t}+\int_0^{\pi}\frac{cost-1}{t}dt)-$$

$$\frac{\ln(2)}{2}(-2)+\ln\left(\frac{1}{2}\right)+\frac{1}{2}=2\ln\left(\frac{\pi}{2}\right)-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}(\ln(\pi)+\ln(\pi)+\int_0^{\pi}\frac{cost-1}{t}dt)+\frac{1}{2}=$$

$$\frac{1}{2}\left(4\ln\left(\frac{\pi}{2}\right)-4Si\left(\frac{\pi}{2}\right)+1\right)-\frac{1}{2}(\ln(\pi)+Ci(\pi)-\gamma)$$

$$\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\frac{1}{2}\left(\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{16}\right)+1\right)$$