ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 (x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2})dx = G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$Let:\ S_{n,k}=\int_0^1 x^{k-1}Li_n(x)dx \overset{I.B.P}{\cong} \frac{x^kLi_n(x)}{k}\Big|_0^1-\frac{1}{k}\int_0^1 x^{k-1}Li_{n-1}(x)dx=\frac{Li_n(1)}{k}-\frac{1}{k}S_{n-1,k}=$$

$$\frac{\zeta(n)}{k}-\frac{1}{k}\left(\frac{\zeta(n-1)}{k}-\frac{1}{k}S_{n-2,k}\right)=\frac{\zeta(n)}{k}-\frac{\zeta(n-1)}{k^2}+\frac{1}{k^2}S_{n-2,k}=\frac{\zeta(n)}{k}-\frac{\zeta(n-1)}{k^2}+\frac{\zeta(n-2)}{k^3}-$$

$$\frac{1}{k^3}S_{n-3,k}=\dots=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+\frac{(-1)^{n-2}}{k^{n-1}}S_{1,k}=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+$$

$$\frac{(-1)^{n-1}}{k^{n-1}}\int_0^1 x^{k-1}\ln(1-x)\,dx=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+\frac{(-1)^{i-1}H_k}{k^n}$$

$$\therefore S_1=\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx$$

$$=\sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^1 x^{2n}ln^2(x)dx \overset{I.B.P}{\cong} \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{2n+1}(x^{2n+1}ln^2(x)-$$

$$\frac{2x^{2n+1}\ln(x)}{2n+1}+\frac{2x^{2n+1}}{(1+2n)^2})\Big|_0^1=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+2)}{(1+2n)^3}=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1+1)}{(1+2n)^3}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^2}+$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^3}=\beta(2)+\beta(3)=G+\frac{\pi^3}{32}$$

$$Then:\ \Omega=\int_0^1(x^{k-1}\sum_{a=1}^{\infty}\frac{x^a}{a^n}+\frac{ln^2(x)}{(1+x^2)^2})dx=\int_0^1 x^{k-1}Li_n(x)dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=$$

$$S_{n,k}+S_1=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+\frac{(-1)^{i-1}H_k}{k^n}+G+\frac{\pi^3}{32}$$

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$$As\ n=3, k=2\ :\ \Omega=\int_0^1 \left(x\sum_{a=1}^{\infty}\frac{x^a}{a^3}+\frac{ln^2(x)}{(1+x^2)^2}\right)dx=\int_0^1 xLi_3(x)dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=$$

$$\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\zeta(4-i)}{2^i}+\frac{H_2}{8}+G+\frac{\pi^3}{32}=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2}\right)dx=\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 x^{n+1}dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=\sum_{n=1}^{\infty}\frac{1}{n^3(n+2)}+$$

$$\left(\frac{ln^2(x)}{2}\left(\frac{x}{x^2+1}+tan^{-1}(x)\right)\right)\Big|_0^1-\int_0^1\frac{\ln(x)}{x^2+1}dx-\int_0^1\frac{\ln(x)\,tan^{-1}(x)}{x}dx=$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n(n+2)}-\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-2}\ln(x)\,dx-$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1 x^{2n-2}\ln(x)\,dx=\frac{\zeta(3)}{2}-\frac{\zeta(2)}{4}+\frac{1}{4}\int_0^1 x\sum_{n=1}^{\infty}\frac{x^n}{n}dx+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}+$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}=\frac{\zeta(3)}{2}-\frac{1}{4}\cdot\frac{\pi^2}{6}-\frac{1}{4}\int_0^1 xln(1-x)dx\ +\ \beta(2)+\beta(3)=\frac{\zeta(3)}{2}-\frac{\pi^2}{24}-$$

$$\frac{1}{4}\cdot\left(2(x^2-1)\ln(1-x)-x(x+2)\right)\Big|_0^1+G+\frac{\pi^3}{32}=\frac{\zeta(3)}{2}-\frac{\pi^2}{24}+G+\frac{\pi^3}{32}+\frac{3}{16}$$

$$\therefore\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2}\right)dx=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2}\right)dx=\int_0^1 x\sum_{n=1}^{\infty}\frac{x^n}{n^3}dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=A+B$$

Working or A:

$$A=\int_0^1 x\sum_{n=1}^{\infty}\frac{x^n}{n^3}dx=\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 x^{n+1}dx=\sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\frac{1}{(n+2)}=\sum_{n=1}^{\infty}\frac{1}{n^3(n+2)}=$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+2}\right]=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{2}\left(\frac{1}{8}\right)=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{16}$$

$$A=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{16}$$

$$\boldsymbol{Working\ or\ B:}$$

$$B=\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx\ \left\{dv=\frac{1}{(1+x^2)^2},\qquad v=\frac{1}{2}\left(\frac{x}{1+x^2}+tan^{-1}(x)\right\}\right.$$

$$\left\{u=ln^2(x),\qquad \frac{du}{dx}=\frac{2\ln(x)}{x}\right\}$$

$$B=\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=\left(\frac{1}{2}\left(\frac{x}{x^2+1}+tan^{-1}(x)\right)ln^2(x)\right)\Big|_0^1-\int_0^1\frac{\ln(x)}{1+x^2}dx-$$

$$\int_0^1\frac{\ln(x)tan^{-1}(x)}{x}dx=-\int_0^1\frac{\ln(x)}{1+x^2}dx-\int_0^1\frac{\ln(x)tan^{-1}(x)}{x}dx=$$

$$-\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n}\ln(x)dx-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1)}\int_0^1x^{2n+1-1}\ln(x)\,dx=$$

$$-\sum_{n=0}^{\infty}(-1)^n\frac{d}{dn}\left(\int_0^1x^{2n}dx\right)-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{d}{dn}\left(\int_0^1x^{2n}dx\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=G+\frac{\pi^3}{32}$$

$$I=A+B=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{16}+G+\frac{\pi^3}{32}=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

$$\therefore\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2}\right)dx=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Solution 4 by Abbaszade Yusif-Azerbaijan

$$\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\left(\frac{\ln x}{1+x^2}\right)^2\right)=\int_0^1\sum_{n=1}^{\infty}\frac{x^{n+1}}{n^3}dx+\int_0^1\left(\frac{\ln x}{1+x^2}\right)^2dx=\xi_1+\xi_2$$

$$\xi_1=\int_0^1\sum_{n=1}^{\infty}\frac{x^{n+1}}{n^3}dx=\sum_{n=1}^{\infty}\frac{1}{n^3(n+2)}=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{n^2-2n+4}{n^3}-\frac{1}{n+2}\right)$$

$$\xi_1=\frac{1}{8}\sum_{n=1}^{\infty}\left(\left(\frac{1}{n}-\frac{1}{n+2}\right)-\frac{2}{n^2}+\frac{4}{n^3}\right)=\frac{1}{8}\left(1+\frac{1}{2}-2\times\frac{\pi^2}{6}+4\zeta(3)\right.$$

$$\xi_1=\frac{1}{8}\left(\frac{3}{2}-\frac{\pi^2}{3}+4\zeta(3)\right)=\frac{3}{16}-\frac{\pi^2}{24}+\frac{\zeta(3)}{2}$$

$$\xi_2=\int_0^1\left(\frac{\ln x}{1+x^2}\right)^2dx=\frac{\ln^2 x}{2}\left(\frac{x}{1+x^2}+\arctan x\right)_0^1-\int_0^1\left(\frac{\ln x}{1+x^2}+\frac{\ln x\arctan x}{x}\right)dx$$

$$\xi_2=-\int_0^1\frac{\ln x}{1+x^2}dx-\ln x\,Ti_2(x)_0^1+\int_0^1\frac{Ti_2(x)}{x}dx\qquad x=\tan(\theta)$$

$$\xi_2=-\int_0^{\frac{\pi}{4}}\frac{\ln(\tan\theta)}{\sec^2\theta}\times\sec^2\theta\,d\theta+Ti_3(1)=\frac{\pi^3}{32}-\int_0^{\frac{\pi}{4}}\ln(tan\,\theta)\,d\theta=\frac{\pi^3}{32}+G$$

$$\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\left(\frac{\ln x}{1+x^2}\right)^2\right)dx=\xi_1+\xi_2=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{16}{3}$$

Note Section :

$$\int_0^x\frac{Ti_{n-1}(t)}{t}dt=Ti_n(x)\,,\quad\int_0^{\frac{\pi}{4}}\ln(tanx)\,dx=-G$$