ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_1^\infty \frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx = G+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)-\frac{3}{4}\ln(2)+\frac{1}{4}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\int_1^\infty \frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx \overset{x\to\frac{1}{x}}{\triangleq} \int_0^1\left(\frac{1}{x^2}-\frac{2}{x}\ln(x)+ln^2(x)\right)\frac{x^2}{(1+x^2)(1+x)^2}dx$$

$$=\int_0^1 \frac{1}{(1+x^2)(1+x)^2}dx-2\int_0^1\frac{xln(x)}{(1+x^2)(1+x)^2}dx+$$

$$\int_0^1\frac{x^2ln^2(x)}{(1+x^2)(1+x)^2}dx \to \quad H=H_1-2H_2+H_3$$

$$H_1=\int_0^1\frac{1}{(1+x^2)(1+x)^2}dx$$

$$=\frac{1}{2}\int_0^1\frac{dx}{x+1}+\frac{1}{2}\int_0^1\frac{dx}{(1+x)^2}-\frac{1}{2}\int_0^1\frac{x}{x^2+1}dx$$

$$=\frac{1}{2}\int_0^1 d\ln(1+x)+\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{4}\int_0^1 dln((1+x))=$$

$$=\frac{\ln(2)}{2}+\frac{1}{2}\cdot\frac{1}{2}-\frac{\ln(2)}{4}=\frac{\ln(2)}{4}+\frac{1}{4}$$

$$H_2=\int_0^1\frac{xln(x)}{(1+x^2)(1+x)^2}dx=$$

$$=\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx$$

$$=\frac{1}{2}\int_0^1\frac{\ln(x)}{(1+x)^2}dx=\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{2k-2}\ln(x)\,dx-\frac{1}{2}\sum_{k\in N}(-1)^{k-1}k\int_0^1 x^{k-1}\ln(x)\,dx=$$

$$-\frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}k}{k^2}=-\frac{\beta(2)}{2}+\frac{1}{2}\int_0^1\sum_{k\in N}(-1)^{k-1}x^{k-1}dx=-\frac{G}{2}+\frac{1}{2}\int_0^1\frac{1}{1+x}dx=$$

$$-\frac{G}{2}+\frac{1}{2}\int_0^1 dl(1+x)=-\frac{G}{2}+\frac{\ln(2)}{2}$$

$$H_3=\int_0^1\frac{x^2ln^2(x)}{(1+x^2)(1+x)^2}dx=\frac{1}{2}\int_0^1\frac{xln^2(x)}{1+x^2}dx$$

$$-\frac{1}{2}\int_0^1\frac{ln^2(x)}{x+1}dx+\int_0^1\frac{ln^2(x)}{(1+x)^2}dx=\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{2k-1}ln^2(x)dx-$$

$$\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{k-1}\,ln^2(x)dx+\frac{1}{2}\sum_{k\in N}(-1)^{k-1}k\int_0^1 x^{k-1}\,ln^2(x)dx$$

$$=\sum_{k\in N}\frac{(-1)^{k-1}}{(2k)^3}-\sum_{k\in N}\frac{(-1)^{k-1}}{k^3}+\sum_{k\in N}\frac{k(-1)^{k-1}}{k^3}=$$

$$-\frac{7}{8}\sum_{k\in N}\frac{(-1)^{k-1}}{k^3}+\sum_{k\in N}\frac{(-1)^{k-1}}{k^2}=-\frac{7}{8}(1-2^{1-3})\zeta(3)+(1-2^{1-2})\zeta(2)=\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)$$

$$H=H_1-2H_2+H_3=\frac{\ln(2)}{4}+\frac{1}{4}-\frac{G}{2}+\frac{\ln(2)}{2}+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)$$

$$=G+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)-\frac{3}{4}\ln(2)+\frac{1}{4}$$

Solution 2 by Ankush Kumar Parcha-India

$$\int_1^\infty\frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_0^1\frac{(1-xln(x))^2}{(1+x^2)(1+x)^2}dx$$

$$=\int_0^1\frac{x^2ln^2(x)}{(1+x^2)(1+x)^2}dx-2\int_0^1\frac{xln(x)}{(1+x^2)(1+x)^2}dx+$$

$$\int_0^1\frac{1}{(1+x^2)(1+x)^2}dx$$

$$=\frac{1}{2}\int_0^1\frac{xln^2(x)}{1+x^2}$$

$$-\frac{1}{2}\int_0^1\frac{ln^2(x)}{x+1}dx$$

$$+\frac{1}{2}\int_0^1\frac{ln^2(x)}{(1+x)^2}dx+\int_0^1\frac{\ln(x)}{(1+x)^2}dx-\int_0^1\frac{\ln(x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{x}{1+x^2}dx+$$

$$\frac{1}{2}\int_0^1\frac{dx}{1+x}+\frac{1}{2}\int_0^1\frac{dx}{(1+x)^2}$$

$$\left(\because\sum_{n\in N}n(-x)^n=-\frac{x}{(1+x)^2},\qquad |x|<1\right)$$

$$\frac{1}{2}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n-1}ln^2(x)dx-\frac{1}{2}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{n-1}ln^2(x)dx+\frac{1}{2}\sum_{n\in N}(-1)^{n-1}n\int_0^1 x^{n-1}ln^2(x)dx+$$

$$+\sum_{n\in N}(-1)^{n-1}n\int_0^1 x^{n-1}\ln(x)\,dx-\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n}\ln(x)\,dx-\int_0^1\frac{d(1+x^2)}{4}+\int_0^1\frac{d(1+x)}{2}$$
$$-\frac{1}{2}\int_0^1 d\left(\frac{1}{1+x}\right)$$

$$(\therefore\int_0^1 t^m ln^n(t)dt=\frac{(-1)^n.\ n!}{(m+1)^{n+1}}\ ,n>-1\bigwedge m\neq 1)$$

$$=\frac{1}{8}\sum_{n\in N}\frac{(-1)^{n-1}}{n^3}-\sum_{n\in N}\frac{(-1)^{n-1}}{n^3}+\sum_{n\in N}\frac{(-1)^{n-1}}{n^2}+\underbrace{\sum_{n\in N}\frac{(-1)^n}{(2n+1)^2}}_{G(Catalan's\ constant)}-\frac{\ln(2)}{4}+\frac{\ln(2)}{2}+\frac{1}{4}$$

$$(\therefore\sum_{n\in N}\frac{x^n}{n}=-\ln(1-x)\,,\qquad |x|\le 1\bigwedge x\neq 1)$$

$$\int_1^\infty\frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx=G+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)-\frac{3}{4}\ln(2)+\frac{1}{4}$$

$G\to Catalan's\ constant$, $\qquad\zeta(3)\to Apery's\ constant$