ROMANIAN MATHEMATICAL MAGAZINE

$$Let: \ a \in R_{\geq 0}\ ; m, n, k, b \ \in Z^{+}$$

$$Find: S = \int_0^1 \left(x^k Li_n(x) + \frac{ln^m(x)}{(1+x^a)^b} \right) dx$$

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$$\int_0^1 \left(x^k Li_n(x) + \frac{ln^m(x)}{(1+x^a)^b} \right) dx = \int_0^1 x^k Li_n(x) dx + \int_0^1 \frac{ln^m(x)}{(1+x^a)^b} dx = I_1 + I_2$$

$$Working\ \ or\ I_1$$

$$I_2 = \int_0^1 \frac{ln^m(x)}{(1+x^a)^b} dx \ \ y = x^a, \qquad x = y^{1/a}, \qquad dx = \frac{dy}{ay^{1-\frac{1}{a}}}$$

$$I_2 = \int_0^1 \frac{ln^m\left(y^{\frac{1}{a}}\right)}{(1+y)^b} \frac{dy}{ay^{1-\frac{1}{a}}} = \frac{1}{a} \int_0^1 \frac{\left(\frac{1}{a}\right)^m}{(1+y)^b y^{1-\frac{1}{a}}} ln^m(y)$$

$$Recall\ that: (1+y)^b = \sum_{t=0}^{\infty} \binom{-b}{t} y^t$$

$$I_2 = \frac{\left(\frac{1}{a}\right)^m}{a} \sum_{t=0}^{\infty} \binom{-b}{t} \int_0^1 y^t \frac{ln^m(y)}{y^{1-\frac{1}{a}}} dy = a^{-m-1} \sum_{t=0}^{\infty} \binom{-b}{t} \int_0^1 y^{t+\frac{1}{a}-1} ln^m(y) dy$$

$$= \left(\frac{1}{a}\right)^{m+1} \sum_{t=0}^{\infty} \binom{-b}{t} \underbrace{\int_0^1 y^{t+\frac{1}{a}-1} ln^m(y)\, dy}_{A}$$

$$Working\ \ on\ \ A$$

$$A = \int_0^1 y^{t+\frac{1}{a}-1} ln^m(y) dy \ \ let: \ln(y) = p \ \ dy = -e^{-p}, \qquad y = e^{-p} \ \ [\infty; 0]$$

$$\int_{\infty}^0 (e^{-p})^{t+\frac{1}{a}-1} (-p)^m . -e^{-p} dp = \int_0^{\infty} (-p)^m (e^{-p})^{\left(t+\frac{1}{a}-1+1\right)} dp = (-1)^m \int_0^{\infty} p^m e^{-p\left(t+\frac{1}{a}\right)} dp$$

$$let\ :\ p\left(t+\frac{1}{a}\right)=x\ \ ;\ p=\frac{x}{t+\frac{1}{a}}\ :dp=\frac{dx}{t+\frac{1}{a}}\quad [0;\infty]$$

$$A=(-1)^m\int_0^\infty \left(\frac{x}{t+\frac{1}{a}}\right)^m e^{-p}\frac{dx}{t+\frac{1}{a}}=\frac{(-1)^m}{(t+\frac{1}{a})^{m+1}}\int_0^\infty x^m e^{-x}dx=\frac{(-1)^m}{(t+\frac{1}{a})^{m+1}}\int_0^\infty x^{m+1-1}e^{-x}dx=$$

$$\frac{(-1)^m}{\left(t+\frac{1}{a}\right)^{m+1}}\Gamma(m+1)\ \ Note\ that:$$

$$\int_0^\infty x^{m-1}e^{-x}=\Gamma(m)\quad \Gamma(m+1)=m\Gamma(m)\quad A=\frac{(-1)^m m\Gamma(m)}{\left(t+\frac{1}{a}\right)^{m+1}}$$

Hence :

$$I_2=\left(\frac{1}{a}\right)^{m+1}\sum_{t=0}^\infty\binom{-b}{t}\underbrace{\int_0^1 y^{t+\frac{1}{a}-1}ln^m(y)\,dy}_{A}\ =\left(\frac{1}{a}\right)^{m+1}\sum_{t=0}^\infty\binom{-b}{t}\frac{(-1)^m m\Gamma(m)}{\left(t+\frac{1}{a}\right)^{m+1}}dy=$$

$$\left(\frac{1}{a}\cdot\frac{1}{t+\frac{1}{a}}\right)^{m+1}\sum_{t=0}^\infty\binom{-b}{t}(-1)^m m\Gamma(m)=\left(\frac{1}{a}\cdot\frac{1}{at+1}\right)^{m+1}\sum_{t=0}^\infty\binom{-b}{t}(-1)^m m\Gamma(m)=$$

$$\left(\frac{1}{at+1}\right)^{m+1}\sum_{t=0}^\infty\binom{-b}{t}(-1)^m m\Gamma(m)=(-1)^m m\Gamma(m)\sum_{t=0}^\infty\binom{-b}{t}\frac{1}{(at+1)^{m+1}}$$

$$I_2=(-1)^m m\Gamma(m)\sum_{t=0}^\infty\binom{-b}{t}\frac{1}{(at+1)^{m+1}}$$

Working on I_1

$$I_1=\int_0^1 x^k Li_n(x)dx\quad u=Li_n(x),\qquad \frac{du}{dx}=\frac{Li_{n-1}(x)}{x},\qquad dv=x^k,\qquad v=\frac{x^{k+1}}{k+1}$$

$$I_1=\int_0^1 x^k Li_n(x)dx$$

$$=\frac{x^{k+1}}{k+1}Li_n(x)\Big|_0^1$$

$$-\int_0^1\frac{Li_{n-1}(x)x^{k+1}}{(k+1)x}dx=\frac{\zeta(n)}{k+1}-\frac{1}{k+1}\left[\frac{Li_{n-1}(x)x^{k+1}}{k+1}\Big|_0^1-\frac{1}{k+1}\int_0^1 x^k Li_{n-2}(x)dx\right]=$$

$$I_1=\frac{\zeta(n)}{k+1}-\frac{\zeta(n)}{(k+1)^2}+\cdots+(-1)^n\frac{\zeta(2)}{(k+1)^{n-1}}+(-1)^{n-1}\frac{H_{k+1}}{(k+1)^n}$$

$$=\sum_{p=1}^{n-1}(-1)^{p-1}\frac{\zeta(n-p+1)}{(k+1)^p}+(-1)^{n-1}\frac{H_{k+1}}{(k+1)^n}$$

$$I=I_1+I_2=\sum_{p=1}^{n-1}(-1)^{p-1}\frac{\zeta(n-p+1)}{(k+1)^p}+(-1)^{n-1}\frac{H_{k+1}}{(k+1)^n}$$

$$+(-1)^m m\Gamma(m)\sum_{t=0}^{\infty}\binom{-b}{t}\frac{1}{(at+1)^{m+1}}$$