ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_1^\infty \frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx$$

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$$I=\int_1^\infty \frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx \overset{\substack{x\to\frac{1}{x}\\ dx=-\frac{dx}{x^2}}}{\cong} \int_1^0 \frac{\ln\left(\frac{1}{x}+1\right)\ln\left(\frac{1}{x^2}+1\right)}{(x+1)^2}-\frac{dx}{x^2}=\int_0^1 \frac{\ln\left(\frac{1+x}{x}\right)\ln\left(\frac{1+x^2}{x^2}\right)}{\frac{(x+1)^2}{x^2}}\frac{dx}{x^2}=$$

$$=\int_0^1 \frac{\ln\left(\frac{1+x}{x}\right)\ln\left(\frac{1+x^2}{x^2}\right)}{(x+1)^2}dx$$

$$\textit{Recall that}: \ln\left(\frac{A}{B}\right)=\ln(A)-\ln(B)$$

$$I=\int_0^1\frac{[\ln(1+x)-\ln(x)][\ln(x^2+1)-\ln(x^2)]}{(x+1)^2}dx=\int_0^1\frac{[\ln(1+x)-\ln(x)][\ln(x^2+1)-2\ln(x)]}{(x+1)^2}dx=$$

$$\int_0^1\frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx-2\int_0^1\frac{\ln(x+1)\ln(x)}{(x+1)^2}dx-\int_0^1\frac{\ln(x)\ln(x^2+1)}{(x+1)^2}dx+2\int_0^1\frac{\ln^2(x)}{(x+1)^2}dx$$

$$\textit{Integration by parts}: \int udv=uv-\int vdu$$

$$I=\left[-\frac{\ln(x+1)+1}{x+1}.\ln(1+x^2)\Big|_0^1+2\int_0^1\frac{x(1+\ln(1+x))}{(1+x)(1+x^2)}dx\right]-2\left[-\frac{\ln(1+x)+1}{1+x}.\ln(x)\Big|_0^1+\int_0^1\frac{\ln(x+1)+1}{x(1+x)}dx\right]-$$

$$\left[\left(\frac{x\ln(x)}{1+x}-\ln(1+x)\right)\ln(1+x^2)\Big|_0^1-2\int_0^1\frac{x^2\ln(x)}{(1+x)(1+x^2)}dx+2\int_0^1\frac{x\ln(1+x)}{1+x^2}dx\right]+$$

$$2\left[\left(\frac{x\ln(x)}{x+1}-\ln(x+1)\right).\ln(x)-\int_0^1\frac{\ln(x)}{x+1}dx+\int_0^1\frac{\ln(x+1)}{x}dx\right]$$

$$I=-\frac{\ln^2(2)}{2}-\frac{\ln(2)}{2}+2\int_0^1\frac{x}{(1+x^2)}dx+2\int_0^1\frac{x\ln(1+x)}{(1+x)(1+x^2)}dx-2\int_0^1\frac{\ln(1+x)}{x(1+x)}dx-2\int_0^1\frac{1}{x(1+x)}dx+\ln^2(2)+$$

$$2\int_0^1\frac{x^2\ln(x)}{(1+x)(1+x^2)}dx-2\int_0^1\frac{x\ln(1+x)}{1+x^2}dx-2\int_0^1\frac{\ln(x)}{x+1}dx+2\int_0^1\frac{\ln(1+x)}{x}dx$$

$$I=\frac{\ln^2(2)}{2}-\frac{\ln(2)}{2}-\int_0^1\frac{dx}{1+x}+\int_0^1\frac{dx}{1+x^2}+\int_0^1\frac{xdx}{1+x^2}-\int_0^1\frac{\ln(x+1)}{1+x}dx+\int_0^1\frac{\ln(x+1)}{1+x^2}dx+\int_0^1\frac{x\ln(x+1)}{1+x^2}dx-$$

$$-2\int_0^1 \frac{\ln(1+x)}{x}dx$$

$$+2\int_0^1 \frac{\ln(1+x)}{1+x}dx$$

$$+2\int_0^1 \frac{dx}{1+x}+\int_0^1 \frac{\ln(x)}{1+x}dx-\int_0^1 \frac{\ln(x)}{1+x^2}dx+\int_0^1 \frac{xln(x)}{1+x^2}dx-2\int_0^1 \frac{xln(1+x)}{1+x^2}dx-$$

$$2\int_0^1 \frac{\ln(x)}{1+x}dx-2\int_0^1 \frac{\ln(1+x)}{x}dx$$

$$This:\ I=\frac{ln^2(2)}{2}-\frac{\ln(2)}{2}-\ln(2)+\arctan(x)\,\begin{matrix}1\\0\end{matrix}+\frac{1}{2}\ln\left(1+x^2\right)\begin{matrix}1\\0\end{matrix}-\frac{1}{2}ln^2(1+x)\,\begin{matrix}1\\0\end{matrix}$$

$$+\int_0^1 \frac{\ln(x+1)}{x^2+1}dx-\int_0^1 \frac{xln(1+x)}{1+x^2}dx-$$

$$4\int_0^1 \frac{\ln(1+x)}{x}dx+ln^2(1+x)\,\begin{matrix}1\\0\end{matrix}+2\ln(1+x)\,\begin{matrix}1\\0\end{matrix}-\int_0^1 \frac{\ln(1+x)}{1+x}dx-\int_0^1 \frac{\ln(x)}{1+x^2}dx+\int_0^1 \frac{xln(x)}{1+x^2}dx$$

$$I=\frac{ln^2(2)}{2}-\frac{\ln(2)}{2}-\ln(2)+\frac{\pi}{4}+\frac{1}{2}\ln(2)-\frac{1}{2}ln^2(2)$$

$$+\int_0^1 \frac{\ln(1+x)}{1+x^2}dx-\int_0^1 \frac{xln(1+x)}{1+x^2}dx-2\int_0^1 \frac{\ln(1+x)}{x}dx+ln^2(2)+$$

$$2ln(2)-\frac{1}{2}ln^2(2)-\int_0^1 \frac{\ln(x)}{1+x^2}dx+\int_0^1 \frac{xln(x)}{1+x^2}dx$$

$$Note\ that:\ \int_0^1 \frac{\ln(1+x)}{1+x^2}dx=\frac{\pi}{8}\ln(2)$$

$$I=\ln(2)+ln^2(2)+\frac{\pi}{4}+\frac{\pi}{8}\ln(2)+\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n-1}\ln(1+x)\,dx+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}dx-$$

$$\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(x)\,dx-\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n-1}\ln(x)\,dx$$

$$I=\ln(2)+\frac{3}{2}ln^2(2)+\frac{\pi}{4}+\frac{\pi}{8}\ln(2)+\sum_{n=1}^{\infty}(-1)^n\left[\frac{\ln(2)}{2n}-\frac{1}{2n}\int_0^1 \frac{x^{2n}}{1+x}dx\right]+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$I=\ln(2)+ln^2(2)+\frac{\pi}{4}+\frac{\pi}{8}\ln(2)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{n}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}+2\left[-\frac{\pi^2}{12}\right]+G-\frac{\pi^2}{48}$$

$$I=G+\ln(2)+\frac{7}{8}ln^2(2)+\frac{5\pi^2}{96}+\frac{\pi}{8}(2+\ln(2))$$