ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_1^{\infty}\frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx$$

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$$I=\int_1^{\infty}\frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx\overset{I.B.P}{\cong}-\frac{\ln(x+1)\ln(x^2+1)}{x+1}\Big|_0^1+2\int_1^{\infty}\frac{x\ln(x+1)}{(x^2+1)(x+1)}dx$$

$$I=\frac{\ln(2)}{2}+2I_1+I_2$$

$$I_1=\int_1^{\infty}\frac{x\ln(x+1)}{(x^2+1)(x+1)}dx\overset{x\to\frac{1}{x}}{\cong}\int_0^1\frac{\ln(x+1)}{(x^2+1)(x+1)}dx-\int_0^1\frac{\ln(x)}{(x^2+1)(x+1)}dx=\frac{1}{2}\int_0^1\frac{(x+1)\ln\left(\frac{2}{x+1}\right)}{x^2+1}dx-$$

$$\frac{1}{2}\int_0^1\frac{\ln(x)}{x^2+1}dx+\frac{1}{2}\int_0^1\frac{x\ln(x)}{x^2+1}dx$$

$$-\frac{1}{2}\int_0^1\frac{\ln(x)}{x+1}dx=\frac{\ln(2)}{2}\int_0^1\frac{x+1}{x^2+1}dx-\frac{1}{2}\int_0^1\frac{(x+1)\ln(x+1)}{x^2+1}dx+\frac{\beta(2)}{2}-\frac{3}{8}\int_0^1\frac{\ln(x)}{x+1}dx$$

$$=\frac{\ln(2)}{2}\left(\frac{\ln(2)}{2}+\frac{\pi}{4}\right)-\frac{1}{2}\int_0^1\frac{x\ln(x+1)}{x^2+1}dx-\frac{1}{2}\int_0^1\frac{\ln(x+1)}{x^2+1}dx+\frac{\beta(2)}{2}+\frac{3}{8}\int_0^1\frac{\ln(x+1)}{x}dx=\frac{ln^2(2)}{4}+\frac{\pi\ln(2)}{8}-$$

$$\frac{1}{2}\left(\frac{\ln(x+1)\ln(x^2+1)}{2}\Big|_0^1\right.$$

$$\left.-\frac{1}{2}\int_0^1\frac{\ln(x^2+1)}{x+1}dx\right)-\frac{1}{2}\int_0^1\frac{\ln\left(\frac{2}{x+1}\right)}{x^2+1}dx+G-\frac{3}{8}\int_0^1 dLi_2(-x)=\frac{ln^2(2)}{4}+\frac{\pi\ln(2)}{8}-\frac{ln^2(2)}{4}+$$

$$\frac{1}{4}\int_0^1\frac{\ln(x^2+1)}{x+1}dx-\frac{\ln(2)}{4}\int_0^1\frac{dx}{x^2+1}+\frac{G}{2}-\frac{3Li_2(-1)}{8}=\frac{\pi\ln(2)}{8}+\frac{1}{4}\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\frac{\ln(2x^2-2\sqrt{2}x+2)}{x}dx-\frac{\pi\ln(2)}{16}+\frac{G}{2}+\frac{\pi^2}{32}$$

$$\left(\therefore\ Li_2(r,\theta)=-\frac{1}{2}\int_0^r\frac{\ln(x^2-2z\cos\theta+1}{z}dz\right)$$

$$=\frac{\pi\ln(2)}{16}+\frac{ln^2(2)}{4}+\frac{1}{4}\left(\frac{1}{2}Li_2\left(\cos^2\left(\frac{\pi}{4}\right)\right)-\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^2\right)+\frac{G}{2}+\frac{\pi^2}{32}=$$

$$=\frac{\pi\ln(2)}{16}+\frac{ln^2(2)}{4}+\frac{1}{4}\left(\frac{1}{2}\left(\frac{\pi^2}{12}-\frac{ln^2(2)}{2}\right)-\frac{\pi^2}{16}\right)+\frac{G}{2}+\frac{\pi^2}{32}$$

$$I_1=\frac{\pi\ln(2)}{16}+\frac{3ln^2(2)}{16}+\frac{5\pi^2}{192}+\frac{G}{2}$$

$$I_2=\int_1^{\infty}\frac{\ln(x^2+1)}{(x+1)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_0^1\frac{\ln(x^2+1)}{(x+1)^2}dx-2\int_0^1\frac{\ln(x)}{(x+1)^2}dx=-\frac{\ln(x^2+1)}{x+1}\Big|_0^1+\int_0^1\frac{x+1}{x^2+1}dx-\int_0^1\frac{dx}{x+1}+2\ln(2)$$

$$I_2 = \frac{3\ln(2)}{2} + \int_0^1 \frac{dx}{x^2+1} - \frac{1}{2}\int_0^1 \frac{dx}{x+1} = \frac{\pi}{4} + \ln(2)$$

$$I = \frac{\ln(2)}{2} + 2I_1 + I_2$$

$$I = \frac{\ln(2)}{2} + 2\left(\frac{\pi\ln(2)}{16} + \frac{3ln^2(2)}{16} + \frac{5\pi^2}{192} + \frac{G}{2}\right) + \frac{\pi}{4} + \ln(2)$$

$$\therefore \int_1^\infty \frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx = G + \frac{5\pi^2}{96} + \frac{7ln^2(2)}{8} + \frac{\pi\ln(2)}{8} + \frac{\pi}{4} + \ln(2)$$

$G \to$ *is the Catalan's constant …*