ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1\int_0^\infty \frac{ln^2(x)ln^2(1+y^2)}{y(1+x^2)}dx=\frac{\pi^3}{64}\zeta(3)$$

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$$We\ \ have;\ \underbrace{\int_0^\infty \frac{ln^2(x)}{(1+x^2)}dx}_{K}\underbrace{\int_0^1 \frac{ln^2(1+y^2)}{y}dy}_{N}(1)$$

$$K=\int_0^\infty \frac{ln^2(x)}{(1+x^2)}dx=\int_0^1 \frac{ln^2(x)}{1+x^2}dx+\underbrace{\int_1^\infty \frac{ln^2(x)}{1+x^2}dx}_{x\to\frac{1}{x}}=2\int_0^1 \frac{ln^2(x)}{1+x^2}dx\overset{\because|x|<1}{\triangleq}$$

$$2\sum_{n\in N\cup\{0\}}(-1)^n\int_0^1 x^{2n}ln^2(x)dx\overset{Note\ Section\ (1)}{\triangleq}4\sum_{n\in N\cup\{0\}}\frac{(-1)^n}{(2n+1)^3}=4\underbrace{\beta(2)}_{\frac{\pi^3}{32}}=4.\frac{\pi^3}{32}=\frac{\pi^3}{8}$$

$$N=\int_0^1 \frac{ln^2(1+y^2)}{y}dy\overset{y^2\to x}{\triangleq}\frac{1}{2}\int_0^1 \frac{ln^2(1+x)}{x}dx=\frac{1}{4}\underbrace{\int_0^1 \frac{1}{x}ln^2\left(\frac{1-x}{1+x}\right)dx}_{\frac{1-x}{1+x}\to x}+\frac{1}{4}\underbrace{\int_0^1 \frac{ln^2(1-x^2)}{x}dx}_{x^2\to x}-$$

$$\frac{1}{2}\underbrace{\int_0^1 \frac{ln^2(1-x)}{x}dx}_{x\to1-x}=\frac{1}{2}\int_0^1 \frac{ln^2(x)}{1-x^2}dx+\frac{1}{8}\underbrace{\int_0^1 \frac{ln^2(1-x)}{x}dx}_{x\to1-x}-\frac{1}{2}\int_0^1 \frac{ln^2(x)}{1-x}dx=$$

$$\frac{1}{2}\int_0^1 \frac{ln^2(x)}{1-x^2}dx-\frac{3}{8}\int_0^1 \frac{ln^2(x)}{1-x}dx=\frac{1}{2}\sum_{n\in N\cup\{0\}}\int_0^1 x^{2n}ln^2(x)dx-\frac{3}{8}\sum_{n\in N}x^{n-1}ln^2(x)dx\overset{Note\ Section\ (1)}{\triangleq}$$

$$\sum_{n\in N\cup\{0\}}\frac{1}{(2n+1)^3}-\frac{3}{4}\sum_{n\in N}\frac{1}{n^3}=\frac{7}{8}\zeta(3)-\frac{3}{4}\zeta(3)=\frac{\zeta(3)}{8}$$

$$\underbrace{\int_0^\infty \frac{ln^2(x)}{(1+x^2)}dx}_{K}\underbrace{\int_0^1 \frac{ln^2(1+y^2)}{y}dy}_{N}=\frac{\pi^3}{8}.\frac{\zeta(3)}{8}=\frac{\pi^3}{64}\zeta(3)$$