ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1 \frac{x\ln(x)}{(x+1)(x^2+1)}dx$$

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Solution by Arowolo Isaiah-Nigeria

$$M = \int_0^1 \frac{x\ln(x)}{(x+1)(x^2+1)}dx$$

$$M = \frac{1}{2}\int_0^1 \frac{x\ln(x)}{x^2+1}dx - \frac{1}{2}\int_0^1 \frac{\ln(x)}{x+1}dx + \frac{1}{2}\int_0^1 \frac{\ln(x)}{x^2+1}dx$$

$$M \overset{x\to x^2}{=} \frac{1}{8}\int_0^1 \frac{\ln(x)}{x+1}dx - \frac{1}{2}\int_0^1 \frac{\ln(x)}{x+1}dx + \frac{1}{2}\int_0^1 \frac{\ln(x)}{x^2+1}dx$$

$$M = -\frac{3}{8}\int_0^1 \frac{\ln(x)}{x+1}dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(tan(x))\,dx$$

$$M = -\frac{3}{8}\left(\int_0^1 \frac{\ln(x)}{1-x}dx + \int_0^1 \frac{2x\ln(x)}{1-x^2}dx\right) + \frac{1}{2}\left(-\frac{1}{2}CI_2\left(\frac{\pi}{2}\right) - \frac{1}{2}CI_2\left(\frac{\pi}{2}\right)\right)$$

$$M = -\frac{3}{8}\int_0^1 \frac{\ln(x)}{1-x}dx + \frac{3}{16}\int_0^1 \frac{\ln(x)}{1-x}dx - \frac{1}{2}CI_2\left(\frac{\pi}{2}\right) \to M = -\frac{3}{16}\int_0^1 \frac{\ln(x)}{1-x}dx - \frac{1}{2}CI_2\left(\frac{\pi}{2}\right)$$

$$M = -\frac{3}{16}\sum_{n=1}^{\infty}\int_0^1 x^{n-1}\ln(x)\,dx - \frac{G}{2} \to \frac{3}{16}\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{G}{2}$$

$$M = \frac{3}{16}\zeta(2) - \frac{G}{2} = \frac{3}{16}\cdot\frac{\pi^2}{6} - \frac{G}{2} = \frac{\pi^2}{32} - \frac{G}{2}$$