ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$I = \int_0^1 \frac{x\log(x)}{(x+1)(x^2+1)}dx = \frac{1}{32}(\pi^2 - 16G)$$

Where, G is Catalan's constant

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Solution by Togrul Ehmedov-Azerbaijan

$$u = \log(x) \to du = dx/x$$

$$dv = \frac{x\log(x)}{(x+1)(x^2+1)}dx \to v = \frac{\tan^{-1}(x)}{2} - \frac{\log(x+1)}{2} + \frac{\log(x^2+1)}{4}$$

$$I \overset{IBP}{=} -\frac{1}{2}\int_0^1 \frac{\tan^{-1}(x)}{x}dx + \frac{1}{2}\int_0^1 \frac{\log(x+1)}{x}dx - \frac{1}{4}\int_0^1 \frac{\log(x^2+1)}{x}dx\Bigg]_{x^2\to x}$$

$$= -\frac{1}{2}\int_0^1 \frac{\tan^{-1}(x)}{x}dx + \frac{1}{2}\int_0^1 \frac{\log(x+1)}{x}dx - \frac{1}{8}\int_0^1 \frac{\log(x+1)}{x}dx =$$

$$= -\frac{1}{2}\int_0^1 \frac{\tan^{-1}(x)}{x}dx + \frac{3}{8}\int_0^1 \frac{\log(x+1)}{x}dx$$

$$= -\frac{1}{2}G + \frac{3}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{k-1}\,dx = -\frac{1}{2}G + \frac{3}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$$

$$= -\frac{1}{2}G + \frac{3}{8}\eta(2) = -\frac{1}{2}G + \frac{3}{16}\varsigma(2) = -\frac{1}{2}G + \frac{\pi^2}{32}$$