ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^1\int_0^1 \frac{x^2\ln(x^2+1)ln^2(y)}{(x^2+1)(y+1)}dxdy$$

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Solution by Izumi Ainsworth-Japan

$$I=\int_0^1\int_0^1 \frac{x^2\ln(x^2+1)ln^2(y)}{(x^2+1)(y+1)}dxdy=\underbrace{\int_0^1\frac{x^2\ln(x^2+1)}{(x^2+1)}dx}_{I_1}\underbrace{\int_0^1\frac{ln^2(y)}{(y+1)}dy}_{I_2}\ldots(\alpha)$$

$$I_1\overset{x^2\to x}{\cong}\frac{1}{2}\int_0^1\frac{\sqrt{x}\ln(x+1)}{x+1}dx=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1 x^{n+\frac{1}{2}}dx=-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{2n+3}=$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_n-\frac{1}{n}\right)}{2n+1}=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{2n+1}-\left(\frac{1}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n(2n+1)}\right)=\underbrace{\beta(2)}_{G}-\frac{\pi\ln(2)}{2}-$$

$$\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}}_{-\ln(2)}+2\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}}_{\frac{\pi}{4}-1}=G-\frac{\pi\ln(2)}{2}+\ln(2)+\frac{\pi}{2}-2\ldots(\beta)$$

$$I_2=\int_0^1\frac{ln^2(y)}{(y+1)}dy=\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^n ln^2(y)dy=2\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}}_{\eta(3)}=\frac{3\zeta(3)}{2}\ldots(\gamma)$$

$$*\ (\beta)\wedge(\gamma)\to(\alpha)$$

$$I=\underbrace{\int_0^1\frac{x^2\ln(x^2+1)}{(x^2+1)}dx}_{I_1}\underbrace{\int_0^1\frac{ln^2(y)}{(y+1)}dy}_{I_2}=\frac{3\zeta(3)}{2}\left(G-\frac{\pi\ln(2)}{2}+\ln(2)+\frac{\pi}{2}-2\right)$$

$$I=\frac{3}{4}\zeta(3)(2G+\pi+\ln(4)-\pi\ln(2)-4)$$